Chern–Connes–Karoubi character isomorphisms and algebras of symbols of pseudodifferential operators

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CHERN-CONNES-KAROUBI CHARACTER ISOMORPHISMS AND ALGEBRAS OF SYMBOLS OF PSEUDODIDIFFERENTIAL OPERATORS

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Abstract. We introduce a class of algebras for which the Chern-Connes-Karoubi character is an isomorphism after tensoring with $\mathbb{C}$. We provide several examples of such algebras, such as invariant sections of Azumaya bundles, crossed products with finite groups and certain algebras of pseudodifferential operators.

Contents

1. Introduction and statement of main results

1.1. **Motivation: invariant operators and their index.** Let G be a compact Lie group acting on a smooth manifold M without boundary. Let $E \to M$ be an equivariant bundle and P be a G-invariant pseudodifferential operator acting on the sections of E. One of the *main motivations* for this paper is to *study the equivariant index* of P using methods of non-commutative geometry [9, 10, 11, 14]. More precisely, we are interested in the index $\mathrm{ind}(\pi_\alpha(P))$ of the restriction $\pi_\alpha(P)$ of the G-invariant operator P to a generic isotypical component $\alpha \in \hat{G}$ [5, 6, 7], because the index provides the main obstruction to the invertibility of these operators. Moreover, the equivariant index $\mathrm{ind}_G(P) \in R(G)$ satisfies

$$\mathrm{ind}_G(P) = \sum_{\alpha \in \widehat{G}} \frac{\mathrm{ind}(\pi_\alpha(P))}{\dim(\alpha)} [\alpha] \in R(G) \,. \tag{1}$$

Let $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$ be the algebra of G-invariant symbols. We may assume, without loss of generality, that P has order zero. Let us consider the K_1^{top}-class $[\sigma_0(P)] \in K_1^{\mathrm{top}}(\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G)$ of the principal symbol $\sigma_0(P) \in \mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$ of P. It is known [10, 14, 15, 17, 33] that the index can be expressed as the pairing $\phi_*([\sigma_0(P)])$ between a cyclic cocycle ϕ on $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$ and the class of $\sigma_0(P)$ in $K_1^{\mathrm{top}}(\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G)$. (We write K^{top} for the *topological* K-theory functors to distinguish them from their algebraic counterparts, see Remark 2.6.) We are thus lead to study the periodic cyclic homology of the symbol algebra $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$. This turns out to be a textbook application of non-commutative geometry using what we dub "Connes algebras," a class of algebras that we introduce and study in this paper.

For a complex algebra A, let

$$\mathrm{Ch} : K_j^{\mathrm{alg}}(A) \to \mathrm{HP}_j^{\mathrm{top}}(A) \,, \quad j = 0, 1 \,, \tag{2}$$

be the Chern-Connes-Karoubi character, where $\mathrm{HP}_j^{\mathrm{top}}(A)$ is a periodic cyclic homology of A defined using a suitable completion for the cyclic complex [10, 11, 23, 13, 12]. Here K_j^{alg} is the algebraic K-theory functor, which is needed since A is not necessarily topological (see Remark 2.6 for its definition and relation to topological K-theory). In order to study the equivariant index of P, we are further led to study whether the map (2) induces isomorphisms

$$\mathrm{Ch} : K_j^{\mathrm{top}}(\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G) \otimes \mathbb{C} \to \mathrm{HP}_j^{\mathrm{top}}(\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G)$$

($j = 0, 1$). Let $\mathcal{C}(S^*M; \mathrm{End}(E))^G$ be the algebra of continuous, G-invariant sections of $\mathrm{End}(E)$, which is, of course, the C^*-completion of $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$. We are also led to study whether the inclusion $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G \subset \mathcal{C}(S^*M; \mathrm{End}(E))^G$ yields isomorphisms in topological K-theory. We prove that the answer to both these questions is affirmative. The techniques for proving these isomorphisms turn out to apply to a more general setting than that of the algebra $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$. We thus introduce and study the class of "Connes algebras," which is, roughly, the class of algebras for which both maps (induced by the character and by inclusion) are isomorphisms. See also [26, 36] for earlier papers that used some similar techniques.

1.2. **Connes algebras and statement of main results.** Let us consider a category of cyclic complexes (this is, roughly, a category of topological algebras for which a definite choice of completion of the cyclic complex has been made, Definition 2.2). We assume that we are given for each $i \in \{0, 1\}$ a suitable K_i-theory functor that is close to topological K_i-theory and is such that the Chern-Connes-Karoubi character (Equation 2) extends to this category. A *Connes algebra* $\mathcal{A}$ for the given K_i-functors is a locally convex topological algebra $\mathcal{A}$ together with a continuous Banach algebra norm $\|\cdot\|_0$ on $\mathcal{A}$ and a given completion of the algebraic cyclic complex of $\mathcal{A}$ with the following properties:

(i) Let $\overline{\mathcal{A}}$ be the completion of $\mathcal{A}$ with respect to the norm $\|\cdot\|_0$. Then the inclusion $\mathcal{A} \to \overline{\mathcal{A}}$ induces an isomorphism in K-theory:

$$K_i(\mathcal{A}) \xrightarrow{\sim} K_i(\overline{\mathcal{A}}), \quad i = 0, 1. \tag{3}$$

(ii) The Chern-Connes-Karoubi character induces an isomorphism "after tensoring with $\mathbb{C}$," that is:

$$\mathrm{Ch} : K_i(\mathcal{A}) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A}), \quad i = 0, 1, \tag{4}$$

where the topological periodic cyclic homology is defined with respect to a suitable completion of the algebraic cyclic complex of $\mathcal{A}$.

See Definition 3.5 for more details.

For the applications in this paper, we shall choose $K_i = \mathrm{RK}_i$, where RK_i were introduced by Phillips [34] (but see also [16, 17] for further insight and important generalizations). The definition of the groups RK_i is recalled below. In this paper, it will be convenient to consider $i \in \mathbb{Z}/2\mathbb{Z} \simeq \{0, 1\}$.

Let X be a manifold with corners (but otherwise $\mathcal{C}^\infty$) and $\mathcal{F} \to X$ be a smooth fiber bundle of finite-dimensional, semi-simple algebras. That is, the fibers of $\mathcal{F}$ are isomorphic to finite direct sums of matrices. We let $\mathcal{C}^\infty(X; \mathcal{F})$ denote the set of smooth sections of $\mathcal{F}$. If $\mathcal{F}$ has simple fibers (i.e. matrix algebras) then this is the prototype of an Azumaya algebra with center $\mathcal{C}^\infty(X)$. We shall need also the following two ideals of $\mathcal{C}^\infty(X; \mathcal{F})$ associated to any closed subset $Y \subset X$. First, $\mathcal{C}_0^\infty(X, Y; \mathcal{F})$ consists of those sections that vanish on Y and, second, $\mathcal{C}_\infty^\infty(X, Y)$ consists of those sections that vanish to *infinite order* on Y. If $\mathcal{F} = \underline{\mathbb{C}}$ (the trivial vector bundle $X \times \mathbb{C} \to X$), then we drop it from the notation.

Theorem 1.1. *Let X be a compact manifold with corners, $\mathcal{F} \to X$ be a smooth fiber bundle of finite-dimensional, semi-simple algebras, and $Y \subset \partial X$ be a union of closed faces of X. We assume that a compact Lie group G acts smoothly on X and $\mathcal{F}$ such that $GY = Y$. Then the algebras $\mathcal{C}^\infty(X; \mathcal{F})^G$, $\mathcal{C}_0^\infty(X, Y; \mathcal{F})^G$, and $\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})^G$ are Connes algebras.*

Essentially the same method of proof leads to the same result when Y is a more general closed subset of X, but that is not needed for our main result and including a proof would greatly extend the length of the paper.

Let $\mathcal{K}$ be the algebra of compact operators. Thus, in particular, the algebra $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G \simeq \Psi^0(M; E)^G / \big(\Psi^0(M; E)^G \cap \mathcal{K}\big)$ is a Connes algebra. To be able to work with the concept of a Connes algebra, we need to make some assumptions that are satisfied in the case of main interest in this paper, that is, that of the algebra $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$. First, we assume that the chosen K-theory functor (the one with respect to which we define the concept of a Connes algebra) is homotopy invariant and satisfies a six term exact sequence for *admissible* short exact sequences of algebras. (The class of admissible exact sequences needs to be specified each time, in our applications, all exact sequences will be admissible as long the ideal and the quotient have the induced topologies. See Sections 2 and 3 for more details.) We also assume that this category satisfies excision in periodic cyclic homology (again for *admissible* short exact sequences) and that the Chern-Connes-Karoubi character is a natural transformation between the exact sequence in K-theory and periodic cyclic homology (associated to, again, an *admissible* short exact sequence of algebras). We then have the following result that will play an important role in our study of the algebras $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$. (In our applications, *all exact sequences will be admissible* if the topologies are compatible.)

Theorem 1.2. *Let $\mathfrak{C}$ be a category of topological cyclic complexes that satisfies excision in periodic cyclic homology and in K-theory. Let $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$ be an admissible short exact sequence in $\mathfrak{C}$ of algebras with a fixed Banach algebra norm such that the corresponding Banach space completion of this sequence is exact as well. Assume that two of the algebras $\mathcal{I}$, $\mathcal{A}$, and $\mathcal{A}/\mathcal{I}$ are Connes algebras. Then the third one is a Connes algebra as well.*

Possible choices of the category $\mathfrak{C}$ and of the K-groups are:

- the category of m-algebras and K-groups of Cuntz with the admissible sequences being the *linear-split* exact sequences [16, 17];
- the subcategory of *Fréchet* m-algebras, in which *all exact sequences are admissible* [30]. This is the case that is relevant for the applications in this paper, and hence this is the case that we shall consider in detail. Moreover, in this case, the K-theory groups of Cuntz coincide with the representable K-theory groups introduced earlier by Phillips [34].

Our result for the algebra $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$ is obtained by iterating Theorem 1.2 for something that we call a C-stratification of ideals (Definition 3.8). More precisely, we have the following result.

Theorem 1.3. *The algebra $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$ is a Fréchet m-algebra with a composition series*

$$0 = I_N \subset I_{N-1} \subset \ldots \subset I_0 = \mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$$

consisting of closed nuclear, Fréchet m-algebra ideals with the following property: For each $j = 0, 1, \ldots, N-1$, there exists a compact manifold with corners Y_j, a smooth bundle $\mathcal{E}_j \to Y_j$ of semisimple algebras and an algebra morphism

$$\phi_j : I_j/I_{j+1} \to \mathcal{C}_0^\infty(Y_j, \partial Y_j; \mathcal{E}_j) := \{f : Y_j \to \mathcal{E}_j \mid f \text{ smooth and } f|_{\partial Y_j} = 0\}$$

that is a homeomorphism onto its image $\phi_j(I_j/I_{j+1})$ and $\cap_{n\in\mathbb{N}}\mathcal{C}_0^\infty(Y_j, \partial Y_j; \mathcal{E}_j)^n \subset \phi_j(I_j/I_{j+1})$.

For $k \in \mathbb{Z}/2\mathbb{Z}$, we let $H^k_{\mathrm{per}}(X, Y) := \oplus_{i\in k} H^i(X, Y) \otimes \mathbb{C}$, that is, the direct sum of all singular cohomology groups of the pair (X, Y) of the same parity as k and with complex coefficients. In particular, as in [26], we have

Corollary 1.4. *Using the notation of Theorem 1.3, the algebra morphisms ϕ_j induce isomorphisms in periodic cyclic homology. Moreover, there is a manifold with corners $\mathfrak{Y}_k$ that is a finite covering $\mathfrak{Y}_k \to Y_k$ such that the center of $\mathcal{C}^\infty(Y_k; \mathcal{E}_k)$ is isomorphic to $\mathcal{C}^\infty(\mathfrak{Y}_k)$ and hence*

$$\mathrm{HP}^{\mathrm{top}}_k(I_j/I_{j+1}) \simeq H^k_{\mathrm{per}}(\mathfrak{Y}_j, \partial\mathfrak{Y}_j)\,, \quad k \in \mathbb{Z}/2\mathbb{Z}\,,$$

and we have a spectral sequence with $E^1_{-p,q} := H^{q-p}_{\mathrm{per}}(\mathfrak{Y}_p, \partial\mathfrak{Y}_p)$ convergent to $\mathrm{HP}^{\mathrm{top}}_{q-p}(\mathcal{C}^\infty(M, \mathrm{End}(E))^G$.

Our algebras are built out of Azumaya algebras, so we include several results proving that they, as well as other algebras related to Azumaya algebras, are Connes algebras. In particular, the commutative algebras modelled by smooth functions are Connes algebras (this is just the initial step in the proof of Theorem 1.1, which is more general).

1.3. **Contents of the paper.** We start is Section 2 with some preliminaries on periodic cyclic homology and topological K-theory. In Section 3, we briefly review excision from an abstract view point and introduce the concepts of Connes algebras, C-smooth stratifications, and C-smooth algebras. We conclude this section by showing that any C-smooth algebra is a Connes algebra. In Section 4, we show that topologically nilpotent algebras are Connes algebras and show that the space of smooth sections $\mathcal{C}_0^\infty(X, Y\mathcal{F})$ of a finite rank bundle $\mathcal{F} \to X$ of semisimple algebras over a manifold with corners X vanishing on a union of closed faces Y is a Connes algebra. This are the first steps for the proof of the main example of Connes algebra discussed in this paper, that is the set of G-invariant smooth sections $\mathcal{C}^\infty(X, Y, \mathcal{F})^G$ of such a bundle $\mathcal{F}$ with $GY = Y$. This example is treated in Section 5 by exhibiting a C-smooth stratification using blow-up constructions. In Section 6, we then deduce from the previous sections that when the group G is finite then the crossed product $\mathcal{C}_0^\infty(X, Y; \mathcal{F}) \rtimes G$ is a Connes algebra, that the algebras of pseudodifferential operators $\Psi^0(M, E)^G$, $\Psi^{-\infty}(M, E)^G$ and $\Psi^{-1}(M, E)^G$, over a closed G-manifold M, with coefficients in a vector bundle E, are Connes algebras. We also show that the algebras of smooth families of pseudodifferential operators $\Psi^0(M|B, E)$, $\Psi^{-\infty}(M|B, E)$ and $\Psi^{-1}(M|B, E)$ are Connes algebras. Appendix A is devoted to the proof of a technical result needed in the proof that $\mathcal{C}^\infty(X, Y, \mathcal{F})^G$ is a Connes algebra.

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2. Background material and preliminary results

For us, a *locally convex topological algebra* is a complex algebra $\mathcal{A}$ endowed with a compatible locally convex topology such that the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous. (The multiplication is thus assumed to be continous jointly in both variables.) In other words, for any continuous semi-norm p on $\mathcal{A}$ there exists a continuous semi-norm p' such that $p(ab) \leq p'(a)p'(b)$, $\forall a, b \in \mathcal{A}$, see [11, Chap 3, Appendix B]. All topological algebras considered in this paper will be assumed to be locally convex.

2.1. **Review of periodic cyclic homology.** We briefly recall the topological periodic cyclic homology for topological algebras that will be used in this paper, see for instance [10, 11, 23, 37] for the standard material used in this section. The topological cyclic homology considered in this paper is such that it is the same for an algebra and for its completion. If $\mathcal{A}$ is a topological algebra, we let $\mathcal{A}^+ := \mathcal{A} \times \mathbb{C}$ denote its unitalization with the product topology and the product $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$. In particular, if $\mathcal{A}$ is a locally convex (complete) topological algebra, then $\mathcal{A}^+$ is equipped with the semi-norms given for instance by $p_\alpha(a,\lambda) := p_\alpha(a) + |\lambda|$. An exact sequence

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0 \tag{5}$$

of locally convex, topological algebras is by definition a sequence of continuous algebra morphisms between locally convex, topological algebras such that $\mathcal{A} \to \mathcal{B}$ is surjective, $\mathcal{B}$ has the induced quotient topology, and $\mathcal{I}$ maps homeomorphically onto its kernel. So it is an algebraically exact sequence with a compatibility of the topologies. In the following, $\mathfrak{A}$ will be a category of locally convex, topological algebras.

Definition 2.1. Let $\mathcal{A}$ be a topological algebra. *A topological cyclic complex on* $\mathcal{A}$ is a graded vector space $\mathcal{C}(\mathcal{A}) := \big(\mathcal{C}_n(\mathcal{A})\big)_{n\geq 0}$, where, for all n, $\mathcal{C}_n(\mathcal{A})$ is a suitable completion of the (algebraic) tensor product $\mathcal{A}^+ \otimes \mathcal{A}^{\otimes n}$. We require that these completions be such that the usual differentials b and B extend by continuity to maps denoted by the same symbols: $b : \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n-1}(\mathcal{A})$ and $B : \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n+1}(\mathcal{A})$, see for instance [10, 11] or [23, Chapter II]. The Hochschild, cyclic, and periodic cyclic homologies of a topological cyclic complex $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ are the corresponding groups defined using the mixed complex $(\mathcal{C}(\mathcal{A}), b, B)$. In particular, the *periodic (topological) cyclic homology groups* $\mathrm{HP}_*^{\mathrm{top}}(\mathcal{A})$ of $\mathcal{A}$ are the homology groups of the complex $\big(\prod_{n\in 2\mathbb{Z}_+} \mathcal{C}_n(\mathcal{A}), \prod_{n\in 2\mathbb{Z}_+} \mathcal{C}_{n+1}(\mathcal{A}), b+B\big)$ ($\mathbb{Z}/2\mathbb{Z}$-graded).

The topological cyclic complexes on (locally convex) topological algebras form a category.

Definition 2.2. Let $\mathfrak{A}$ be a category of locally convex algebras with continuous algebra morphisms, as above. A *category of topological cyclic complexes* (based on the objects of $\mathfrak{A}$) is a category $\mathfrak{C}$ whose objects are pairs $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ consisting of topological algebra $\mathcal{A}$ in $\mathfrak{A}$ together with a topological cyclic complex $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$. The morphisms in $\mathfrak{C}$ are continuous, linear maps $f = (f_a, f_c) : (\mathcal{A}, \mathcal{C}(\mathcal{A})) \to (\mathcal{B}, \mathcal{C}(\mathcal{B}))$ such that $f_a : \mathcal{A} \to \mathcal{B}$ is an algebra morphism in $\mathfrak{A}$ and $f_c(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = f_a(a_0) \otimes f_a(a_1) \otimes \ldots \otimes f_a(a_n)$.

2.2. **Review of topological K-theory.** In order to define a topological K-functor satisfying the properties which are necessary for our study of Connes algebras (such as periodicity) some further conditions need to be imposed on the chosen category of locally convex algebras, see for instance [11, 16, 34]. In our applications, we will use the Cuntz category of m-algebras. We have therefore devoted this brief review to the topological K-functor for Fréchet m-algebras, as introduced and studied by Phillips in [34]. For general m-algebras, we refer the interested reader to the excellent survey [17].

Recall that a given complete topological algebra $\mathcal{A}$ as above is an m-algebra when its topology can be defined by a family of submultiplicative semi-norms [16, 17, 31]. A Fréchet m-algebras is an m-algebra which is Fréchet, in other words, it is a complete topological algebra whose topology is defined by a countable family of submultiplicative semi-norms. Notice that each of these two categories is stable under projective tensor products. A first important example of a Fréchet m-algebra is the algebra $\mathcal{R}$ of infinite complex matrices with rapidly decreasing entries, see [11, 16, 34]. An element $R = (R_{ij})_{i,j\in\mathbb{N}} = \sum_{i,j\in\mathbb{N}} R_{ij} E_{ij}$ in $\mathcal{R}$ ($R_{ij} \in \mathbb{C}$) satisfies:

$$p_n(R) := \sum_{i,j\in\mathbb{N}} (1+i+j)^n |R_{ij}| < +\infty, \quad \forall \in \mathbb{N}\,.$$

Given any complete topological algebra $\mathcal{A}$, the completed projective tensor product algebra $\mathcal{R}\hat{\otimes}\mathcal{A}$ will be denoted $\mathcal{R}\mathcal{A}$. It is an m-algebra when $\mathcal{A}$ is an m-algebra [17]. It is a Fréchet algebra when $\mathcal{A}$ is a Fréchet algebra [34]. Recall that any continuous homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of locally convex topological algebras extends to the unitalizations as well as to a homomorphism of locally convex topological algebras $M_N(\mathcal{A}) \to M_N(\mathcal{B})$, $N \in \mathbb{N}$, and $\mathcal{R}\mathcal{A} \to \mathcal{R}\mathcal{B}$.

We also consider the topological algebra $\mathcal{C}^\infty([a,b])\hat{\otimes}\mathcal{A} \simeq \mathcal{C}^\infty([a,b];\mathcal{A})$ of smooth $\mathcal{A}$-valued functions on $[a,b]$. As in the Introduction, the algebra of complex smooth functions on $[0,1]$ which vanish with all their derivatives at 0 and 1 will be denoted $\mathcal{C}^\infty_\infty([0,1],\{0,1\})$. We the define the *smooth suspension* $\mathcal{S}\mathcal{A}$ of the topological algebra $\mathcal{A}$ by $\mathcal{S}\mathcal{A} := \mathcal{C}^\infty_\infty([0,1],\{0,1\})\hat{\otimes}\mathcal{A}$. Again, if $\mathcal{A}$ is an m-algebra (respectively, a Fréchet m-algebra) then so are all the above tensor products.

We recall next the definiton of the groups RK_0 and RK_1 introduced by Phillips [34], but we formulate it in the more general setting of locally convex topological algebras.

Definition 2.3. Let $\mathcal{A}$ be a locally convex topological algebra.

(1) Two idempotents $e_0, e_1 \in \mathcal{A}$ will be called *smoothly homotopic* if there exists an idempotent $e \in \mathcal{C}^\infty([0,1],\mathcal{A})$ such that $e(0) = e_0$ and $e(1) = e_1$.
(2) Denote by $\bar{P}(\mathcal{A})$ the set of idempotents $e \in M_2((\mathcal{R}\mathcal{A})^+)$ such that $e - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{R}\mathcal{A})$. Then the set of smooth homotopy classes in $\bar{P}(\mathcal{A})$ is denoted $\mathrm{RK}_0(\mathcal{A})$.
(3) Denote by $\bar{U}(\mathcal{A})$ the set of invertible elements $u \in (\mathcal{R}\mathcal{A})^+$ such that $u - 1 \in \mathcal{R}\mathcal{A}$. Then the set of smooth homotopy classes in $\bar{U}(\mathcal{A})$ is denoted $\mathrm{RK}_1(\mathcal{A})$.

If $\mathcal{A}$ is a Fréchet m-algebra, Phillips has shown that $\mathrm{RK}_0(\mathcal{A})$ and $\mathrm{RK}_1(\mathcal{A})$ are abelian groups that depend functorially on $\mathcal{A}$ [34, 35]. This result obviously remains true in the category of locally convex topological algebras, however, the category of Fréchet m-algebras has several additional nice properties. In the following, the term "excision" is used in the usual sense, which is recalled in Definition 2.4. In the following $\mathfrak{A}$ will be a category of locally convex algebras with continuous algebra morphisms for which we shall assume that there is given a distinguished class of exact sequences of $\mathfrak{A}$ called *admissible exact sequences*.

Definition 2.4. Let $\mathfrak{A}$ be a category whose objects are locally convex algebras and whose morphisms are (suitable) continuous algebra morphisms. We shall say that $\mathfrak{A}$ *satisfies the excision property for* RK if, for any admissible short exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ in $\mathfrak{A}$, there exist natural boundary maps $\mathrm{RK}_0(\mathcal{B}) \to \mathrm{RK}_1(\mathcal{I})$ and $\mathrm{RK}_1(\mathcal{B}) \to \mathrm{RK}_0(\mathcal{I})$ that, together with the functorial morphisms, yield a six-term exact sequence:

$$\begin{array}{ccccc} \mathrm{RK}_0(\mathcal{I}) & \longrightarrow & \mathrm{RK}_0(\mathcal{A}) & \longrightarrow & \mathrm{RK}_0(\mathcal{B}) \\ \uparrow & & & & \downarrow \\ \mathrm{RK}_1(\mathcal{B}) & \longleftarrow & \mathrm{RK}_1(\mathcal{A}) & \longleftarrow & \mathrm{RK}_1(\mathcal{I})\,. \end{array}$$

Theorem 2.5 (Phillips)**.** *The representable K-theory functors* RK_0 *and* RK_1 *satisfy stability, Bott periodicity, and excision (or six term short exact sequence) on the category of Fréchet m-algebras (in which all exact sequences are admissible).*

Remark 2.6. In addition to the RK_i functors, we shall also use the *algebraic* K-theory functors $K_i^{\mathrm{alg}}(A)$, $i = \{0,1\}$ defined as the Grothendieck group of the semi-group of finitely generated, projective modules over the unital algebra A, for $i = 0$, respectively as the limit of the abelianizations of $GL_n(A)$, as $n \to \infty$, for $i = 1$. If A is a topological algebra with unit, we shall also use the *topological* K-theory functors $K_i^{\mathrm{top}}(A)$, which are the quotient of their algebraic counterparts with respect to homotopy. For a non-unital algebra A, $K_n(A)$ is the kernel of $K_n(A^+) \to K_n(\mathbb{C})$, where K denotes any of the K-functors considered. We have natural surjections $K_i^{\mathrm{alg}}(A) \to K_i^{\mathrm{top}}(A)$. If A happens to be a Banach algebra, then $K_0^{\mathrm{alg}}(A) \simeq K_0^{\mathrm{top}}(A)$, but, in general, $K_1^{\mathrm{alg}}(A) \to K_1^{\mathrm{top}}(A)$ is not injective. We also have natural maps $K_i^{\mathrm{top}}(A) \to \mathrm{RK}_i(A)$ if A is a locally convex topological algebra, which are isomorphisms if A is a Banach algebra or if it is a subalgebra of a Banach algebra that is stable under holomorphic functional calculus, see [11, Chapter 3, Appendix C, Proposition 3].

Remark 2.7. Recall the Chern-Connes-Karoubi character $\mathrm{Ch} : K_i^{\mathrm{alg}}(A) \to \mathrm{HP}_i^{\mathrm{alg}}(A)$, defined for any complex algebra [10, 11, 23]. This definition extends right away to the RK-functors. Indeed, let $\mathfrak{C}$ be a category of topological cyclic complexes and $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ be an object of $\mathfrak{C}$. Then the natural map of complexes $\mathcal{C}^{\mathrm{alg}}(\mathcal{A}) := (A^{\otimes(n+1)})_{n\in\mathbb{Z}_+} \to \mathcal{C}(\mathcal{A})$ will induce a natural group morphism $\mathrm{HP}_*^{\mathrm{alg}}(\mathcal{A}) \to \mathrm{HP}_*^{\mathrm{top}}(\mathcal{A})$ such that Ch descends to a map $\mathrm{Ch} : (K_i^{\mathrm{alg}}(A)/\sim) \to \mathrm{HP}_i^{\mathrm{top}}(A)$, $i = 0, 1$ where $\sim$ denotes the smooth homotopy of projections or of invertible elements. As a consequence, we obtain a Chern-Connes-Karoubi character $\mathrm{Ch} : \mathrm{RK}_i(\mathcal{A}) \to \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A})$ extending the classical Chern-Connes-Karoubi character on algebraic K-theory:

$$\mathrm{RK}_i(\mathcal{A}) \to \big(K_i^{\mathrm{alg}}(\mathcal{R}\widehat{\otimes}\mathcal{A})/\sim\big) \xrightarrow{\mathrm{Ch}} \mathrm{HP}_i^{\mathrm{top}}(\mathcal{R}\widehat{\otimes}\mathcal{A}) \xrightarrow{Tr_*} \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A}) .$$

3. Connes' principle and main results

We introduce a class of algebras for which the Chern-Connes-Karoubi character is an isomorphism (after tensoring with $\mathbb{C}$). Several examples of such algebras will be provided in Sections 4, 5, and 6 in relation with algebras of symbols of certain pseudodifferential operators.

3.1. Categories with the excision property for HP and for RK.

We shall treat excision from an abstract view point. In the following, we will have to specify each time which are the "admissible" exact sequences that we consider.

Definition 3.1. Let $\mathfrak{A}$ be a category of locally convex topological algebras in which a class of exact sequences, called *admissible* is given. Let $\mathfrak{C}$ be a category of topological cyclic complexes. We shall say that $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ is *an admissible exact sequence* in $\mathfrak{C}$ if it is an admissible exact sequence of topological algebras in $\mathfrak{A}$, $\mathcal{C}(\mathcal{B})$ is the quotient of $\mathcal{C}(\mathcal{A})$ under the induced morphism and $\mathcal{C}(\mathcal{I}) \to ker\big(\mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})\big)$ is a homeomorphism onto its image.

In particular, the cyclic complexes of $\mathcal{B}$ and $\mathcal{I}$ are determined by that of $\mathcal{A}$.

Definition 3.2. We shall say that a category $\mathfrak{C}$ of topological cyclic complexes *satisfies the excision property for periodic cyclic homology* (or for HP) if for any object $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ of $\mathfrak{C}$ and for any admissible short exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ in $\mathfrak{C}$ (see Definition 3.1) the natural inclusion $\mathcal{C}(\mathcal{I}) \to \ker\big(\mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})\big)$ induces an isomorphism of the corresponding periodic cyclic homology groups.

It is known that the category of Fréchet m-algebras satisfies excision, but, for general m-algebras, the result is known only for *split* exact sequences [17, 18, 19, 30].

Remark 3.3. A consequence of the excision property of Definition 3.2 is that there exist boundary maps $\mathrm{HP}_0^{\mathrm{top}}(\mathcal{B}) \to \mathrm{HP}_1^{\mathrm{top}}(\mathcal{I})$ and $\mathrm{HP}_1^{\mathrm{top}}(\mathcal{B}) \to \mathrm{HP}_0^{\mathrm{top}}(\mathcal{I})$ that, together with the functorial morphisms, yield a six-term exact sequence in periodic cyclic homology:

$$\begin{array}{ccccc} \mathrm{HP}_0^{\mathrm{top}}(\mathcal{I}) & \longrightarrow & \mathrm{HP}_0^{\mathrm{top}}(\mathcal{A}) & \longrightarrow & \mathrm{HP}_0^{\mathrm{top}}(\mathcal{B}) \\ \uparrow & & & & \downarrow \\ \mathrm{HP}_1^{\mathrm{top}}(\mathcal{B}) & \longleftarrow & \mathrm{HP}_1^{\mathrm{top}}(\mathcal{A}) & \longleftarrow & \mathrm{HP}_1^{\mathrm{top}}(\mathcal{I})\,. \end{array}$$

Thus Phillips' Theorem 2.5 in particular states that the category of *Fréchet m-algebras* satisfies the excision property for RK. See also [17]. In the same way, the results of Cuntz state that if $\mathfrak{C}$ is the larger category of m-algebras with $K_* := kk_*(\mathbb{C}, \bullet)$ (where kk is the Cuntz bivariant K-theory functor) then this category also satisfies the K-excision property. We shall need the following extension of a result in [33].

Proposition 3.4. *Let $\mathfrak{C}$ be a category of topological cyclic complexes that satisfies excision for* HP *and for* RK *and let $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$ be a short exact sequence in $\mathfrak{C}$. Assume that for two of the algebras $\mathcal{I}$, $\mathcal{A}$, and $\mathcal{B}$, the Chern-Connes-Karoubi character* $\mathrm{Ch} : \mathrm{RK}_i \otimes \mathbb{C} \to \mathrm{HP}_i^{\mathrm{top}}$ *is an isomorphism, then* Ch *is an isomorphism also for the third algebra.*

Proof. The results in [33] and Cuntz [16] imply that Ch is a natural transformation from the six-term exact sequence of Definition 2.4 to the six-term exact sequence of Remark 3.3. The result then follows from the Five Lemma [4]. □

See also [22, 36].

3.2. **Connes algebras.** Recall that a *Banach algebra-norm* $\|\cdot\|_0$ on a topological algebra $\mathcal{A}$ is a continuous norm on $\mathcal{A}$ such that $\|ab\|_0 \le \|a\|_0 \|b\|_0$.

Definition 3.5. Let $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ be a topological cyclic complex on $\mathcal{A}$, where $\mathcal{A}$ is an algebra endowed with a continuous Banach norm $\|\cdot\|_0$. Let $\overline{\mathcal{A}}$ be the completion of $\mathcal{A}$ in the norm $\|\cdot\|_0$. Then $(\mathcal{A}, \mathcal{C}(\mathcal{A}), \|\cdot\|_0)$ is called a *Connes algebra* if, for all $j \in \mathbb{Z}/2\mathbb{Z}$, the following two maps

(i) $\mathrm{RK}_j(\mathcal{A}) \to \mathrm{RK}_j(\overline{\mathcal{A}})$ and
(ii) $\mathrm{Ch} : \mathrm{RK}_j(\mathcal{A}) \otimes \mathbb{C} \to \mathrm{HP}_j^{\mathrm{top}}(\mathcal{A})$

are isomorphisms.

The isomorphism (i) and the Chern-Connes-Karoubi character

$$\mathrm{Ch} : \mathrm{RK}_n(\mathcal{A}) \longrightarrow \mathrm{HP}_n^{\mathrm{top}}(\mathcal{A}) \tag{6}$$

then yield Connes' character

$$\widetilde{\mathrm{Ch}} : K_n(\overline{\mathcal{A}}) \longrightarrow \mathrm{HP}_n^{\mathrm{top}}(\mathcal{A})\,. \tag{7}$$

We are ready now to prove Theorem 1.2.

Proof of Theorem 1.2. Item (i) of the definition of Connes algebra (Definition 3.5) is a direct consequence of the six terms exact sequences and a standard diagram chase. Item (ii) of that definition was proved in Proposition 3.4. □

We are interested in Connes algebras because of "Connes' principle", which can be stated as follows:

Theorem 3.6 (Connes' principle)**.** *If $\mathcal{A}$ is a Connes algebra with $\overline{\mathcal{A}}$ its completion with respect to the given norm $\|\cdot\|_0$, then Connes' character $\widetilde{\mathrm{Ch}} : K_i(\overline{\mathcal{A}}) \otimes \mathbb{C} \to \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A})$ is an isomorphism, for all $i \in \mathbb{Z}/2\mathbb{Z}$.*

Connes' principle follows immediately from the definition of Connes algebras and the preceding discussion. Moreover, it permeates his earlier works on cyclic homology and it is what sets them appart from other related works on cyclic homology. It is, of course, due to Connes. It is a useful principle since the K-groups of C^*-algebras are notoriously difficult to compute.

Remark 3.7. Let us gather some basic observations about the class of Connes algebras.

(1) The class of Connes algebra is stable with respect to direct sums.
(2) The algebra $\mathcal{C}^\infty(M)$ of smooth functions over a closed manifold M is a Connes algebra with respect to the sup-norm. This follows directly from Connes' result [10]. Such algebras are basically the building blocks of the class of Connes algebras.
(3) Moreover, for any (Fréchet) Connes m-algebra $\mathcal{A}$, the (Fréchet) m-algebra $\mathcal{C}^\infty(M, \mathcal{A})$ is a Connes algebra for the sup-norm associated with the given Banach norm on $\mathcal{A}$. See [34] for item (i) of the definition of Connes algebras. Item (ii) follows from Künneth formulas in cyclic homology, see [24, 28].
(4) In the same way, the (Fréchet) m-algebra $\mathcal{RA}$ is a Connes algebra when $\mathcal{A}$ is a Connes algebra. One can use many Banach completions of $\mathcal{RA}$, see [17]. See also [34] for the case of Fréchet m-algebras.

In view of definition 3.5, we now introduce the *Banach version* $\mathfrak{C}_0$ of a category of cyclic complexes $\mathfrak{C}$ as the category of triples $(\mathcal{A}, \mathcal{C}(\mathcal{A}), \|\cdot\|_0)$ where $(\mathcal{A}, \mathcal{C}(\mathcal{A}))$ is an objet in $\mathfrak{C}$ (a locally convex topological algebra $\mathcal{A}$ together with a choice of completion of its cyclic mixed complex $\mathcal{C}(\mathcal{A})$) and a continuous Banach algebra norm on $\mathcal{A}$. The morphisms in $\mathfrak{C}_0$ are the morphisms in $\mathfrak{C}$ that are continuous with respect to the given Banach algebra norms. The main result of this paper states that the category of Connes algebras behaves well with respect to exact sequences. For simplicity, we shall assume from now on that all exact sequences in $\mathfrak{A}$ are admissible. This is the case for Fréchet m-algebras, which is the case that is used in our main result.

Definition 3.8. Let $\mathfrak{C}$ be a category of topological cyclic complexes that satisfies excision for both HP and RK and let $(\mathcal{A}, \mathcal{C}(\mathcal{A}), \|\cdot\|_0)$ be an object in its Banach version $\mathfrak{C}_0$.

(1) A *C-smooth stratification* of $\mathcal{A}$ is a sequence of two-sided ideals of $\mathcal{A}$ in $\mathfrak{C}$,
$$0 = \mathcal{I}_N \subset \mathcal{I}_{N-1} \subset \ldots \subset \mathcal{I}_1 \subset \mathcal{I}_0 = \mathcal{A},$$
such that $\mathcal{I}_k/\mathcal{I}_{k+1}$ are Connes algebras for the Banach norms induced from $\mathcal{A}$.

(2) The topological algebra $\mathcal{A}$ is a C-smooth algebra if it admits such a C-smooth stratification.

We shall also write $I_j := \overline{\mathcal{I}}_j$. We are ready now to state the main result of this section.

Theorem 3.9. *Every C-smooth algebra $\mathcal{A}$ in a category that satisfies excision in K-theory and in periodic cyclic homology is a Connes algebra. In particular, the Connes character $\widetilde{\mathrm{Ch}} : K_i^{\mathrm{top}}\left(\overline{\mathcal{A}}\right) \otimes \mathbb{C} \longrightarrow \mathrm{HP}_i^{\mathrm{top}}(\mathcal{A})$ is an isomorphism.*

Proof. We shall proceed by induction on N. If $N = 0$ or 1, there is nothing to prove. Assume then that $N \geq 2$ and that the result is known for $\mathcal{I}_1$ and let us prove it for $\mathcal{A} = \mathcal{I}_0$. The completion of the exact sequence

$$0 \to \mathcal{I}_1 \to \mathcal{I}_0 \to \mathcal{I}_0/\mathcal{I}_1 \to 0 \tag{8}$$

with respect to the given Banach space norms is exact since we have assumed that the norms are induced from $\mathcal{A}$. Moreover, this is an exact sequence in $\mathfrak{C}$ since we have assumed that I_1 is an ideal of I_0 in $\mathfrak{C}$ and that $\mathfrak{C}$ satisfies the excision property for HP. We know that $\mathcal{I}_0/\mathcal{I}_1$ is a Connes algebra by the hypothesis. We also know that $\mathcal{I}_1$ is a Connes algebra by the induction hypothesis. Theorem 1.2 then gives that $\mathcal{I}_0$ is also a Connes algebra. □

4. Some examples of Connes algebras

We already mentioned many standard examples of Connes algebras and we now list some others that will be used in the next section in the study of the algebra of G-equivariant symbols on manifolds. The standard Goodwillie argument plays an important part in the applications and computations, see [21]. Recall that this argument is based on the vanishing of the periodic cyclic homology for the class of (topologically) nilpotent algebras. These algebras appear naturally when dealing with appropriate quotients of ideals, and form, in some sense, a class of "negligible" Connes algebras, since their K-theory groups as well as their periodic cyclic spaces are trivial. Let us restrict ourselves again to the category of Fréchet m-algebras (and hence, all exact sequences will be admissible). Also, from now on in this paper, the category $\mathfrak{C}$ will be the category of Fréchet m-algebras with the cyclic complex defined using the projective tensor product. The category $\mathfrak{C}_0$ is the subcategory of $\mathfrak{C}$ of algebras equipped with a fixed Banach algebra norm. The morphisms are the morphisms in $\mathfrak{C}$ that are continuous with respect to the fixed norm. The exact sequences are supposed to yield exact sequences of Banach algebras (for the induced norms).

Definition 4.1. [Meyer] A complete locally convex topological algebra $\mathcal{N}$ will be called a *topologically nilpotent algebra,* if, for any semi-norm p on $\mathcal{N}$, there exists $k \geq 1$, such that $p(\mathcal{N}^k) = 0$ (that is, $p(f_1 \dots f_k) = 0$ for any $f_1, \dots, f_k \in \mathcal{N}$).

Notice that such a topologically nilpotent algebra cannot be unital.

Proposition 4.2. *Let $\mathcal{N}$ be a topologically nilpotent Fréchet m-algebra with a Banach norm $\|\cdot\|_0$. Then $\mathcal{N}$ is spectrally invariant in its Banach completion $\overline{\mathcal{N}}$ with respect to $\|\cdot\|_0$. Moreover, all groups $\mathrm{HP}_i(\mathcal{N})$, $K_0^{\mathrm{alg}}(\mathcal{N})$, $K_i^{\mathrm{top}}(\mathcal{N})$, $\mathrm{RK}_i(\mathcal{N})$, $K_0^{\mathrm{alg}}(\overline{\mathcal{N}})$, $K_i^{\mathrm{top}}(\overline{\mathcal{N}})$, and $\mathrm{RK}_i(\overline{\mathcal{N}})$, $i \in \mathbb{Z}/2\mathbb{Z}$, vanish and hence $\mathcal{N}$ is a Connes algebra.*

Proof. It is well known that the topological periodic cyclic homology of any topologically nilpotent algebra $\mathcal{N}$ is trivial. For the topological periodic cyclic homology, we refer for instance to [21, 29, 30]. As for the vanishing of K-theory groups, this is easy to check. First, for any $x \in M_N(\mathcal{N})$, the power series $\sum_{k=0}^{\infty} x^k$ converges, and hence $1 - x$ is invertible. This shows that $K_1^{\mathrm{top}}(\mathcal{N}) = 0$. Let $p_0 \in M_N(\mathbb{C})$ and let $e \in M_N((\mathcal{N})^+)$ be any idempotents which satisfies that $e - p_0 \in M_N(\mathcal{N})$. Then $u := ep_0 + (1 - e)(1 - p_0)$ satisfies $eu = up_0$ and is an invertible element in $M_N(\mathcal{N}^+)$. Hence e and p_0 are equivalent. This proves that $K_0^{\mathrm{alg}}(\mathcal{N}) = 0$, and hence that $K_0^{\mathrm{top}}(\mathcal{N}) = 0$, again since the later is the quotient of the former, see Remark 2.6.

Finally, let $\overline{\mathcal{N}}$ be some Banach completion of $\mathcal{N}$. Next, both $\mathcal{R}\mathcal{N}$ and $\overline{\mathcal{N}}$ are also topologically nilpotent. The same arguments gives then $\mathrm{RK}_0(\mathcal{N}) = \mathrm{RK}_1(\mathcal{N}) = 0$. They also show that $K_0^{\mathrm{alg}}(\overline{\mathcal{N}}) = 0 = K_0^{\mathrm{top}}(\overline{\mathcal{N}})$ and $K_1^{\mathrm{top}}(\overline{\mathcal{N}}) = 0$. □

Let us deduce some consequences for the algebras of smooth functions on compact manifolds with corners. Let X be a compact manifold with corners. Recall that, for a closed subspace $Y \subset X$, $\mathcal{C}_0^{\infty}(X, Y)$ is the ideal in $\mathcal{C}^{\infty}(X)$ consisting of those complex valued smooth functions that vanish on Y and that $\mathcal{C}_{\infty}^{\infty}(X, Y)$ is the set of those smooth complex valued functions that *vanish to infinite order* on Y. It is an ideal of $\mathcal{C}_0^{\infty}(X, Y)$.

We shall need Azumaya bundles over compact manifolds with corners.

Notation 4.3.
- (i) $\mathcal{S}, \mathcal{S}_i \to X$ are smooth, bundles of finite dimensional, *simple* algebras.
- (ii) $\mathcal{F} \to X$ is a smooth bundle of finite dimensional, *semi-simple* algebras.
- (iii) $E \to X$ is a smooth vector bundle on X.
- (iv) $\mathcal{C}^{\infty}(X; E)$ is the vector space of smooth sections of E and $\mathcal{C}(X; \mathcal{F})$ is the vector space of *continuous* sections of $\mathcal{F}$.
- (v) Let $Y \subset X$ be a closed subset, then $\mathcal{C}_0(X \smallsetminus Y; E)$ is the vector space of *continuous* sections of $\mathcal{F}$ that *vanish* on Y, $\mathcal{C}_0^{\infty}(X, Y; E)$ is the vector space of smooth sections of E that *vanish* on Y, and $\mathcal{C}_{\infty}^{\infty}(X, Y; E)$ is the vector space of smooth sections of E that *vanish to infinite order* on Y.

So $\mathcal{C}^{\infty}(X)$, $\mathcal{C}_0^{\infty}(X, Y)$, and $\mathcal{C}_{\infty}^{\infty}(X, Y)$ coincide respectively with $\mathcal{C}^{\infty}(X; E)$, $\mathcal{C}_0^{\infty}(X, Y; E)$, and $\mathcal{C}_{\infty}^{\infty}(X, Y; E)$ for $E = \underline{\mathbb{C}}$, the one dimensional trivial bundle.

The fibers of $\mathcal{S}$ and $\mathcal{S}_i$ are matrix algebras (one block), so they have natural C^*-norms. The fibers of $\mathcal{F}$ are direct sums of matrix algebras, so they also have natural C^*-norms. In particular, locally we have $\mathcal{F} \simeq \oplus_{i=1}^{k} \mathcal{S}_i$, but *not* globally. Similarly, we have locally that $\mathcal{S} \simeq \mathrm{End}(E)$, but not globally. In fact, every point x of X has an open neighborhood U such that $\mathcal{S}|_U \simeq U \times M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. In particular, this allows us to define natural C^*-norms on the spaces of smooth sections, which by completion give rise to continuous sections.

The algebra $\mathcal{C}^{\infty}(X, \mathcal{F})$ is a prototype of an Azumaya algebra (over its center) and in Proposition 4.5 will show that it is a Connes algebra. Recall that a unital Fréchet m-algebra $\mathcal{A}$ is an Azumaya algebra over its center $\mathcal{Z}$ if $\mathcal{A}$ is a finitely generated projective module over Z such that

$$\mathcal{A} \otimes_{\mathcal{Z}} \mathcal{A}^{op} = \mathrm{End}_{\mathcal{Z}}(\mathcal{A}),$$

where $\mathcal{A}^{op}$ is the algebra $\mathcal{A}$ with the opposite product and $\mathrm{End}_{\mathcal{Z}}(\mathcal{A})$ is the algebra of continuous $\mathcal{Z}$-linear maps on $\mathcal{A}$. We shall denote by $\mathrm{Prim}(A)$ the *Primitive ideal* *spectrum* of a C^*-algebra A, that is, the set of primitive ideals of A with the hull-kernel topology [20]. (Recall that an ideal $I \subset A$ is primitive if, by definition, it is the kernel of a non-zero irreducible *-representation of A.)

Lemma 4.4. *Let X be a compact manifold with corners and let $\mathcal{F} \to X$ be a locally trivial bundle of semisimple algebras as above. Let Y be a closed subspace of X. Set $\mathcal{A} := \mathcal{C}^\infty(X, \mathcal{F})$, $\mathcal{I} := \mathcal{C}_0^\infty(X, Y; \mathcal{F})$, and let A and I be their completions in the "sup-norm."*

(1) *Let $\mathfrak{X} := \mathrm{Prim}(A)$. Then $\mathfrak{X}$ is a finite covering of X and hence a manifold with corners.*
(2) *The center $\mathcal{Z}$ of $\mathcal{A}$ is isomorphic to $\mathcal{C}^\infty(\mathfrak{X})$. Moreover, $\mathcal{Z} \simeq \mathcal{C}^\infty(X, Z(\mathcal{F}))$, where $Z(\mathcal{F}) \subset \mathcal{F} \to X$ is the center of $\mathcal{F}$.*
(3) *The algebra $\mathcal{A}$ is an Azumaya algebra over $\mathcal{Z}$, more precisely, $\mathcal{A} \simeq \mathcal{C}^\infty(\mathfrak{X}; \mathcal{F}_A)$, where $\mathcal{F}_A$ is a bundle of* simple *algebras over $\mathfrak{X}$, canonically obtained from $\mathcal{F}$.*
(4) *The algebra $A \simeq \mathcal{C}(\mathfrak{X}; \mathcal{F}_A)$ is an Azumaya algebra over its center $Z \simeq \mathcal{C}(\mathfrak{X})$.*

Proof. We may assume X to be connected. On each trivialization U of $\mathcal{F}$, we have $\mathcal{F}|_U \simeq U \times \bigoplus_{k=1}^p M_{n_k}(\mathbb{C})$. This implies that the primitive ideal spectrum $\mathfrak{X} := \mathrm{Prim}(A)$ of the C^*-completion $A := \mathcal{C}(X, \mathcal{F})$ of $\mathcal{A}$ is a covering of X. Hence $\mathfrak{X}$ has the structure of a manifold with corners. Next, the center $\mathcal{Z}$ of $\mathcal{A}$ coincides with the bundle over X with fibers given by the centers of the fibers of $\mathcal{F}$. It is isomorphic to the algebra $\mathcal{C}^\infty(\mathfrak{X})$ as can be checked locally.

For the third item, let $\mathcal{F}_A \to \mathfrak{X}$ be the fiber bundle of simple algebras given by $\mathcal{F}_A|_{U \times \{k\}} \simeq U \times M_{n_k}(\mathbb{C})$, with the transition functions induced by $\mathcal{F}$. Then $\mathcal{A} \simeq \mathcal{C}^\infty(\mathfrak{X}, \mathcal{F}_\mathcal{A})$ and hence the $\mathcal{Z}$-module $\mathcal{A}$ is projective and finitely generated. The proof is completed by a local inspection. Recall that if we denote by $\mathbb{C}^p$ the center of $\bigoplus_{k=1}^p M_{n_k}(\mathbb{C})$, then we have

$$(\oplus_{k=1}^p M_{n_k}(\mathbb{C})) \otimes_{\mathbb{C}^p} (\oplus_{k=1}^p M_{n_k}(\mathbb{C}))^{op} \cong \mathrm{End}_{\mathbb{C}^p}(\oplus_{k=1}^p M_{n_k}(\mathbb{C}))\,.$$

The last statement is thus a direct consequence of (3). □

The following proposition is well-known for X smooth without boundary (or corners) [10, 11, 23, 22]. Its proof generalizes right away to manifolds with corners and to sections of semi-simple bundles.

Proposition 4.5. *Let X be a compact manifold with corners, $\mathcal{F} \to X$ be a fiber bundle of semi-simple algebras over X, and $\mathcal{Z} \simeq \mathcal{C}^\infty(\mathfrak{X})$ be the center of $\mathcal{A} = \mathcal{C}^\infty(X, \mathcal{F})$. Then the inclusion $i : \mathcal{Z} \to \mathcal{A}$ induces isomorphisms $\mathrm{HP}_n(\mathcal{Z}) \to \mathrm{HP}_n(\mathcal{A})$ and $\mathrm{RK}_n(\mathcal{Z}) \otimes \mathbb{Q} \to \mathrm{RK}_n(\mathcal{A}) \otimes \mathbb{Q}$, $n \in \mathbb{Z}/2\mathbb{Z}$. Let Z and A be the norm completions of these algebras. Then, similarly, the inclusion $Z \subset A$ induces isomorphisms $\mathrm{RK}_n(Z) \otimes \mathbb{Q} \to \mathrm{RK}_n(A) \otimes \mathbb{Q}$, $n \in \mathbb{Z}/2\mathbb{Z}$. Consequently, $\mathcal{Z}$ and $\mathcal{A}$ are Connes algebras with*

$$\mathrm{HP}_n(\mathcal{A}) \simeq \mathrm{HP}_n(\mathcal{Z}) \simeq H^n_{\mathrm{per}}(\mathfrak{X})\,.$$

Proof. First, let us prove that if $\mathcal{F} = \underline{\mathbb{C}} := X \times \mathbb{C}$, then $\mathcal{C}^\infty(X; \mathcal{F}) = \mathcal{C}^\infty(X)$ is a Connes algebra as in the case without corners. We follow the classical proof (see [10, 11, 22, 23] for the justifications not included here). Let $i \in \mathbb{Z}/2\mathbb{Z}$. since $\mathcal{C}^\infty(X)$ is stable under holomorphic calculus in $\mathcal{C}(X)$, we have that $K_i^{\mathrm{top}}(\mathcal{C}^\infty(X)) \to K_i^{\mathrm{top}}(\mathcal{C}(X))$ is an isomorphism. Since $\mathcal{C}(X)$ is a C^*-algebra, we also have $\mathrm{RK}_i(\mathcal{C}^\infty(X)) \simeq K_i^{\mathrm{top}}(\mathcal{C}^\infty(X)) \simeq K_i^{\mathrm{top}}(\mathcal{C}(X)) \simeq \mathrm{RK}_i(\mathcal{C}(X)) \simeq K^i(X)$. Next, because X has the homotopy type of a finite CW-complex and Ch is a natural transformation of cohomology theories that is an isomorphism for spheres, we also have that $\mathrm{Ch} : \mathrm{RK}_i(\mathcal{C}(X)) \otimes \mathbb{C} \simeq K^i(X) \otimes \mathbb{C} \to H^i_{\mathrm{per}}(X)$ is an isomorphism. Finally, we have that the classical proof of $\mathrm{HP}_i(\mathcal{C}^\infty(X)) \simeq H^i_{\mathrm{per}}(X)$ (using the Hochschild-Kostant-Rosenberg-Connes isomorphism) applies without change. We have thus obtained the sequence of isomorphisms

$$\mathrm{RK}_i(\mathcal{C}^\infty(X))\otimes\mathbb{C} \simeq K_i^{\mathrm{top}}(\mathcal{C}(X))\otimes\mathbb{C} \simeq K^i(X)\otimes\mathbb{C} \simeq H^i_{\mathrm{per}}(X) \simeq \mathrm{HP}_i(\mathcal{C}^\infty(X)) \tag{9}$$

whose composition is the Chern-Connes-Karoubi map, which is hence an isomorphism. This proves that $\mathcal{C}^\infty(X)$ is a Connes algebra.

The center $\mathcal{Z}$ of $\mathcal{A}$ is $\mathcal{Z} \simeq \mathcal{C}^\infty(\mathfrak{X})$. By Proposition 4.5, by replacing X with $\mathfrak{X}$ and $\mathcal{F}$ with $\mathcal{F}_A$, we may assume that $\mathcal{F}$ consists of simple fibers (i.e. matrix algebras) and hence that $\mathcal{A}$ is Azumaya over X. In particular, we may replace $\mathfrak{X}$ with X in what follows. Let us consider then the natural embeddings:

$$\mathcal{Z} \xhookrightarrow{\ i\ } \mathcal{A} \xhookrightarrow{\ j\ } \mathrm{End}_Z(\mathcal{A}) \xhookrightarrow{\ k\ } M_N(\mathcal{Z}) \xhookrightarrow{\ l\ } M_N(\mathcal{A})\,,$$

where the morphism k comes from an embedding $\mathcal{F} \subset X \times \mathbb{C}^N$ of vector bundles over X. It is well known that $\mathrm{HP}_n(M_N(\mathcal{Z})) \simeq \mathrm{HP}_n(\mathcal{Z})$ and that $\mathrm{RK}_n(M_N(\mathcal{Z})) \simeq \mathrm{RK}_n(\mathcal{Z})$ and that the same results hold for $\mathcal{A}$. We thus obtain the compositions $k_* \circ j_* \circ i_* : K^n(X) \to K^n(X)$ and $l_* \circ k_* \circ j_* : K_n^{\mathrm{top}}(\mathcal{A}) \to K_n^{\mathrm{top}}(\mathcal{A})$. The first morphism is multiplication with the class $[\mathcal{A}] \in K^0(X)$, which is an invertible element in $K^0(X) \otimes \mathbb{Q}$. Similarly, $K_*^{\mathrm{top}}(\mathcal{A})$ is a module over $K^*(X)$ using the isomorphism $\mathcal{Z} \otimes_{\mathcal{Z}} \mathcal{A} \simeq \mathcal{A}$ and $l \circ k \circ j$ is obtained from $k \circ j \circ i$ by tensoring with $id_{\mathcal{A}}$ over $\mathcal{Z}$. Hence $l_* \circ k_* \circ j_*$ is also multiplication by $[\mathcal{A}]$. It follows that both compositions $k_* \circ j_* \circ i_*$ and $l_* \circ k_* \circ j_*$ become isomorphisms after tensoring with $\mathbb{Q}$ and hence $i_* : \mathrm{RK}_n(\mathcal{Z}) \to \mathrm{RK}_n(\mathcal{A})$ is an isomorphism. The same argument gives that the inclusion $Z \to A$ induces an isomorphism $\mathrm{RK}_n(Z) \otimes \mathbb{Q} \to \mathrm{RK}_n(A) \otimes \mathbb{Q}$.

To obtain the corresponding result in periodic cyclic homology, we first notice that we have also maps $\mathrm{HH}_n(i) : \mathrm{HH}_n(\mathcal{Z}) \to \mathrm{HH}_n(\mathcal{A})$ of Hochschild homology groups that can be localized using the $\mathcal{Z}$-module structure on both groups. Once we localize, the same argument as above (really just Morita equivalence) gives that all localizations to maximal ideals in $\mathcal{Z}$ of $\mathrm{HH}_n(i)$ are isomorphisms, and hence $\mathrm{HH}_n(i)$ is an isomorphism as well. (This is the argument from [26].) Hence i induces an isomorphism in periodic cyclic homology as well. Since we have proved that $\mathcal{Z} \simeq \mathcal{C}^\infty(\mathfrak{X}) = \mathcal{C}^\infty(X)$ is a Connes algebra, it follows that $\mathcal{A}$ is a Connes algebra as well (since, up to tensoring with $\mathbb{C}$, it has the same RK and HP groups). □

The following result will be a basic step in the proofs of the results of the next section.

Proposition 4.6. *Let $Y \subset X$ be a closed subspace and $\mathcal{I}$ be a closed subalgebra of $\mathcal{C}_0^\infty(X, Y; \mathcal{F})$ that contains $\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})$. Then, for all $i \in \mathbb{Z}/2\mathbb{Z}$, the natural inclusions induce isomorphisms:*

(1) $\mathrm{RK}_i(\mathcal{C}_0^\infty(X, Y; \mathcal{F})) \simeq \mathrm{RK}_i(\mathcal{C}_0(X \smallsetminus Y; \mathcal{F}))$;
(2) $\mathrm{RK}_i(\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})) \simeq \mathrm{RK}_i(\mathcal{I})$;
(3) $\mathrm{RK}_i(\mathcal{I}) \simeq \mathrm{RK}_i(\mathcal{C}_0^\infty(X, Y; \mathcal{F}))$;
(4) $\mathrm{HP}_i^{\mathrm{top}}(\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})) \simeq \mathrm{HP}_i^{\mathrm{top}}(\mathcal{I})$; *and*
(5) $\mathrm{HP}_i^{\mathrm{top}}(\mathcal{I}) \simeq \mathrm{HP}_i^{\mathrm{top}}(\mathcal{C}_0^\infty(X, Y; \mathcal{F}))$.

Proof. The algebra $\mathcal{C}_0^\infty(X,Y,\mathcal{F})$ is stable under holomorphic functional calculus in $\mathcal{C}_0(X\smallsetminus Y,\mathcal{F})$, and hence the inclusion induces an isomorphism $\mathrm{RK}_i(\mathcal{C}_0^\infty(X,Y,\mathcal{F}))\to K_i^{\mathrm{top}}(\mathcal{C}_0(X\smallsetminus Y,\mathcal{F}))$. The Fréchet m-algebra $\mathcal{C}_0^\infty(X,Y,\mathcal{F})$ can be endowed with the uniform norm and is an object in the category $\mathfrak{C}_0$ whose completion is precisely the C^*-algebra $\mathcal{C}_0(X\smallsetminus Y,\mathcal{F})$. The same observation holds for the smaller subalgebra $\mathcal{C}_\infty^\infty(X,Y,\mathcal{F})$, which is also a Fréchet m-algebra. In particular, we have an isomorphism $\mathrm{RK}_i(\mathcal{C}_\infty^\infty(X,Y,\mathcal{F}))\to K_i^{\mathrm{top}}(\mathcal{C}_0(X\smallsetminus Y,\mathcal{F}))$.

Now notice that the quotient Fréchet m-algebra $\mathcal{I}/\mathcal{C}_\infty^\infty(X,Y,\mathcal{F})$ is topologically nilpotent when endowed with the quotient topology. Indeed, the semi-norms on the quotient $\mathcal{C}_0^\infty(X,Y,\mathcal{F})/\mathcal{C}_\infty^\infty(X,Y,\mathcal{F})$, can be taken to be a sequence induced by the standard (semi-)norms on $\mathcal{C}_0^\infty(X,Y,\mathcal{F})/\mathcal{C}_\ell^\infty(X,Y,\mathcal{F})$, where $\mathcal{C}_\ell^\infty(X,Y,\mathcal{F})$ is the algebra of smooth functions vanishing of order $\leq \ell$ on Y. Therefore since any such semi-norm on $\mathcal{C}_0^\infty(X,Y,\mathcal{F})/\mathcal{C}_\ell^\infty(X,Y,\mathcal{F})$ vanishes on ℓ-products, we conclude that $\mathcal{C}_0^\infty(X,Y,\mathcal{F})/\mathcal{C}_\infty^\infty(X,Y,\mathcal{F})$ is topologically nilpotent as claimed. This also gives that $\mathcal{I}/\mathcal{C}_\infty^\infty(X,Y,\mathcal{F})$ is topologically nilpotent. Hence $\mathrm{HP}_i(\mathcal{I}/\mathcal{C}_\infty^\infty(X,Y,\mathcal{F}))=0=\mathrm{RK}_i(\mathcal{I}/\mathcal{C}_\infty^\infty(X,Y,\mathcal{F}))$. Then, excision in K-theory and in topological periodic cyclic homology for the short exact sequence

$$0\to\mathcal{C}_\infty^\infty(X,Y,\mathcal{F})\hookrightarrow\mathcal{I}\longrightarrow\mathcal{I}/\mathcal{C}_\infty^\infty(X,Y,\mathcal{F})\to 0$$

of Fréchet m-algebras gives that $\mathrm{RK}_i(\mathcal{C}_\infty^\infty(X,Y;\mathcal{F}))\to\mathrm{RK}_i(\mathcal{I})$ and $\mathrm{HP}_i^{\mathrm{top}}(\mathcal{C}_\infty^\infty(X,Y;\mathcal{F}))\to\mathrm{HP}_i^{\mathrm{top}}(\mathcal{I})$ are isomorphisms for all $i\in\mathbb{Z}/2\mathbb{Z}$. By replacing $\mathcal{I}$ with $\mathcal{C}_0^\infty(X,Y;\mathcal{F})$, we obtain that $\mathrm{RK}_i(\mathcal{C}_\infty^\infty(X,Y;\mathcal{F}))\to\mathrm{RK}_i(\mathcal{C}_0^\infty(X,Y;\mathcal{F}))$ and $\mathrm{HP}_i^{\mathrm{top}}(\mathcal{C}_\infty^\infty(X,Y;\mathcal{F}))\to\mathrm{HP}_i^{\mathrm{top}}(\mathcal{C}_0^\infty(X,Y;\mathcal{F}))$ are isomorphisms. Hence the inclusion $\mathcal{I}\to\mathcal{C}_0^\infty(X,Y;\mathcal{F})$ gives that the maps $\mathrm{RK}_i(\mathcal{I})\to\mathrm{RK}_i(\mathcal{C}_0^\infty(X,Y;\mathcal{F}))$ and $\mathrm{HP}_i^{\mathrm{top}}(\mathcal{I})\to\mathrm{HP}_i^{\mathrm{top}}(\mathcal{C}_0^\infty(X,Y;\mathcal{F}))$ are also isomorphisms. This completes the proof. □

Theorem 4.7. *Let X be a compact manifold with corners and $Y\subset\partial X$ be a union of closed faces of X. Let $\mathcal{I}$ be a closed subalgebra of $\mathcal{C}_0^\infty(X,Y;\mathcal{F})$ containing $\mathcal{C}_\infty^\infty(X,Y;\mathcal{F})$. Then the algebras $\mathcal{C}_\infty^\infty(X,Y;\mathcal{F})$, $\mathcal{I}$, and $\mathcal{C}_0^\infty(X,Y;\mathcal{F})$ are all Connes algebras with respect to the uniform norm of $\mathcal{C}_0(X\smallsetminus Y,\mathcal{F})$.*

Proof. We shall assume for simplicity that $\mathcal{F}=\underline{\mathbb{C}}$, the general case being the same using that $\mathcal{C}^\infty(X,\mathcal{F})$ is a Connes algebra, see Proposition 4.5. Recall that the boundary depth of a boundary face is the number of vanishing coordinates in a corner chart $\mathbb{R}^n\times[0,\infty)^k$, see [3] for the precise definition. We shall proceed by induction on the dimension of X. If $\dim X=0$ then $\partial X=\emptyset$ and the cardinal of X is finite say k. Thus $\mathcal{C}^\infty(X)\cong\mathbb{C}^k$, which is trivially a Connes algebra. Similarly, if $\dim X=1$ then ∂X and hence also $Y=\{x_1,\cdots,x_k\}$ are a finite sets. Applying Theorem 1.2 to the exact sequence

$$0\to\mathcal{C}_0^\infty(X,Y)\to\mathcal{C}^\infty(X)\to\mathbb{C}^k\to 0\,,$$

we get that $\mathcal{C}_0^\infty(X,Y)$ is a Connes algebra.

Now assume that the statment is true for smaller dimensions than that of X, that is, that the algebras $\mathcal{C}_0^\infty(Z,Z_Y)$ are Connes algebras for all compact manifolds with corners Z such that $\dim Z<\dim X$ (and $Z_Y\subset\partial Z$ a union of closed faces of Z). Denote by N the maximal depth of X, i.e. $N=\max\operatorname{depth}(F)$, where F runs over all the boundary faces. Let us introduce the ideals $\mathcal{I}_k:=\mathcal{C}_0^\infty(X,Y\cap\cup_{\operatorname{depth}(F)\leq k}F)$ and consider the stratification

$$\mathcal{I}_N=\mathcal{C}_0^\infty(X,Y)\subset\mathcal{I}_{N-1}\subset\cdots\subset\mathcal{I}_1\subset\mathcal{I}_0=\mathcal{C}^\infty(X).$$

Then all quotients are given by

$$\mathcal{I}_k/\mathcal{I}_{k+1} \cong \bigoplus_{F\subset Y, \mathrm{depth}(F)=k+1} \mathcal{C}_0^\infty(F, Y\cap \partial F).$$

Indeed, the restriction map $r : \mathcal{I}_k \to \bigoplus_{F\subset Y,\mathrm{depth}(F)=k+1} \mathcal{C}_0^\infty(F, Y \cap \partial F)$ is clearly well defined and has kernel exactly $\mathcal{I}_{k+1}$. Moreover, r is surjective because, if $f \in \mathcal{C}_0^\infty(F, Y \cap \partial F)$, then we can extend f to a smooth function $\tilde{f}$ on X that vanishes on the other faces $\subset Y$ of the same depth as F since the surjectivity is a local statement (using a partition of unity) and, locally, we may assume that F is an embedded boundary face and hence that it has a tubular neighborhood. That is, we can assume that $\tilde{f}$ is equal to 0 on any boundary face contained in Y distinct from F and of depth $k + 1$.

Since $\mathcal{C}_0^\infty(F, Y\cap\partial F)$ are Connes algebras by the induction hypothesis, we obtain that $\mathcal{I}_k/\mathcal{I}_{k+1}$ is a Connes algebra as finite direct sum of Connes algebras. Now, $\mathcal{C}^\infty(X)$ is a Connes algebra and all quotients $\mathcal{I}_k/\mathcal{I}_{k+1}$ are Connes algebras thus the above stratification is C-smooth and applying Theorem 1.2 several times, we deduce that $\mathcal{I}_k$ is a Connes algebra for all k. In particular, $\mathcal{I}_N = \mathcal{C}_0^\infty(X, Y)$ is a Connes algebra.

The statements about $\mathcal{I} \subset \mathcal{C}_0^\infty(X, Y)$ is a consequence of Proposition 4.2 since $\mathcal{C}_0^\infty(X, Y)/\mathcal{I}$ is topologically nilpotent (it is also a consequence of Proposition 4.6). The statement about $\mathcal{C}_\infty^\infty(X, Y)$ is a particular case. □

5. Group actions on Azumaya bundles

The goal of this section is to prove Theorem 5.9.

5.1. *G*-manifolds with corners.

We shall use the notation from [25] (page 4 and pages 49-50) for transformation groups and the notation from [3] for manifolds with corners and blow-ups. See also [1, 27]. The reader should also consult these references for the missing definitions or proofs. The definition of the blow-up $[X : Y]$ is recalled in the Appendix, see Equation (23).

We let X be a compact manifold with corners with a smooth G-action, We may assume that X is endowed with a smooth Riemannian metric and that the action of G is isometric. We shall consider *submanifolds* of X in the strong sense that they have tubular neighborhoods. Thus, if $Y \subset X$ is a *submanifold with corners* of X, then Y is closed and there is a neighborhood U of Y in X such that U is G-diffeomorphic to the normal vector bundle $N^XY := TX|_Y/TY \to Y$ (the normal vector bundle to Y in X) via a diffeomorphism of manifolds with corners mapping the zero section of N^XY to Y. This is the concept of manifolds with corners considered in [2]. In particular, a submanifold with corners of X is also a p-submanifold of X [3, 27], but the converse is not true. A submanifold with corners is called *an interior p-submanifold* in [1].

If $x \in X$, G_x denotes the isotropy group of x, namely,

$$G_x := \{\gamma \in G \mid \gamma(x) = x\}. \qquad (10)$$

Given a subgroup $H \subset G$, we let (H) denote the set of subgroups of G conjugated to H. If $K \subset G$ is another subgroup, we will write $(H) \leq (K)$ if H is conjugated to a subgroup of K. Since X is compact, we know that there exist only finitely many conjugacy classes $C_j = (H_j) = (G_{x_j})$, $j = 1, \ldots, N$, of isotropy groups G_x, $x \in X$.

Then $C_j \leq C_k$ if there is a subgroup in C_j that is contained in a subgroup in C_k. We may assume the order (numbering) to be such that,

$$C_j \leq C_k \Longrightarrow j \geq k.$$

An order with these properties will be called an *admissible* order.

In the following, H and K will denote subgroups of G. We shall use the following standard notations [25]

$$\begin{aligned} X_H &:= \{x \in M \mid G_x = H\}\,, \\ X(H) &:= \{x \in X \mid (G_x) = (H)\}\,, \\ X(\geq H) &:= \{x \in X \mid (G_x) \geq (H)\}\,, \quad \text{and} \\ X(\leq H) &:= \{x \in X \mid (G_x) \leq (H)\}\,. \end{aligned} \tag{11}$$

As usual, X^H denotes the set of fixed points by H. It is known then that

$$GX^H \;=\; X(\geq H) \;\text{ and }\; GX_H \;=\; X(H)\,. \tag{12}$$

We also denote by $N(H) := \{g \in G \mid g^{-1}Hg = H\}$ the normalizer of H in G. Moreover, $N(H)$ acts on X^H and we have the following diffeomorphism that will be used later on:

$$X(H) \simeq (G/H) \times_{N(H)/H} X_H\,, \tag{13}$$

with the induced action of $N(H)/H$ on X_H (Proposition 1.91 and Corollaries 1.92 and 1.94 of [25]). Moreover, the action of $N(H)/H$ on X_H is free. See [25] for the following obvious lemma.

Lemma 5.1. *We have $\overline{X(H)} \subset X(\geq H)$. Thus, if H is a maximal isotropy group, then $X(H)$ is closed.*

As in [1], we shall say that the action of G on X is *boundary intersection free* if, given a closed face F of M and $g \in G$, we have either $gF = F$ or $gF \cap F = \emptyset$. Notice that if the action of G on X is boundary intersection free, then so is the action of any subgroup $H \subset G$. The first part of the following statement is a result from [1].

Lemma 5.2. *Assume that the action of G on X is boundary intersection free. Then X^G is a (closed) submanifold with corners of X and $\partial[X^G] = [\partial X]^G = X^G \cap \partial X$. Similarly, $X(G)$ is a manifold with corners and $\partial[X(G)] = [\partial X](G) = X(G) \cap \partial X$. If $H \subset G$ is a maximal isotropy subgroup, then $X(H)$ will be a submanifold with corners of X. If H is not a maximal isotropy subgroup, then $X(H)$ will still be a manifold with corners, but not closed, in general, and hence not a submanifold with corners of X, in general.*

Proof. Let us fix a G-invariant metric on X. Let $x \in X^G$. Let us suppose that x belongs to an open face F of X and let us choose a G-invariant neighborhood U of x in F. By using the exponential map in directions normal to F, we obtain then that x has a G-invariant neighborhood of the form $U \times [0,1)^j$ with the action of G being diagonal and trivial on $[0,1)^j$ since G is compact and its action on X is boundary intersection free. Hence $(U \times [0,1)^j)^G = U^G \times [0,1)^j$ and U^G is a smooth manifold without boundary. This proves that X^G is a manifold with corners. We further have that

$$\partial\big[(U\times[0,1)^j)^G\big] \;=\; \partial\big[U^G\times[0,1)^j\big] \;=\; U^G\times\partial\big[[0,1)^j\big] \;=\; \Big(U\times\partial\big[[0,1)^j\big]\Big)^G \;=\; \big(\partial(U\times[0,1)^j)\big)^G\,.$$

This proves that $\partial[X^G] = [\partial X]^G = X^G \cap \partial X$, since this is a local property. By choosing a tubular neighborhood of U^G in U, we obtain a tubular neighborhood of $U^G \times [0,1)^j$ in $U \times [0,1)^j$ and hence that X^G is a submanifold with corners. The other statements are proved in a similar way by using that $U(H)$ is an immersed submanifold of U that is closed if H is a maximal isotropy subgroup (Lemma 5.1). □

Lemma 5.2 allows us to unambiguously write for any closed subgroup of G

$$\partial X^H = \partial[X^H] = [\partial X]^H = X^H \cap \partial X \text{ and } \partial X(H) = \partial[X(H)] = [\partial X](H) = X(H) \cap \partial X.$$

That is "$\partial(\cdot)$ commutes with $\cdot^H$ and with $\cdot(H)$."

Remark 5.3. The statement of the last lemma is a local statement, so no assumption of having embedded faces is required.

We formulate the following result as a lemma, for the purpose of further referencing it. Except the statement about V_1 (which follows from Lemma 5.2), it is Lemma 1.81 of [25].

Lemma 5.4. *Assume again that the action of G on the compact manifold with corners X is boundary intersection free, and let $\{H_1, H_2, \ldots, H_N\}$ be a complete set of representatives of conjugacy classes of isotropy groups with an admissible ordering (that is, $(H_i) \geq (H_j)$ implies $i \leq j$). For $1 \leq k \leq N$, we set*

$$V_k := X(H_1) \cup X(H_2) \cup \ldots \cup X(H_k). \tag{14}$$

Then V_k is a closed subset of X, but not a submanifold, in general, except V_1, which is a submanifold with corners of X.

The subspaces $X(H_j)$, $j = 1, \ldots, N$ define a stratification of X.

5.2. **G-equivariant bundles and algebras.** Recall the objects introduced in the notation 4.3, objects that from now on will be endowed with an action of our compact Lie group G. In particular, $\mathcal{S}, \mathcal{S}_i \to X$ will be G-equivariant bundles of finite dimensional *simple* algebras, and $\mathcal{F} \to X$ will be a G-equivariant bundle of finite dimensional, *semi-simple* algebras. Given a closed subset $Y \subset X$ with $GY = Y$, and a G-equivariant vector bundle E, recall also the ideals $\mathcal{C}_0^\infty(X, Y; E)$ and $\mathcal{C}_\infty^\infty(X, Y; E)$ of 4.3, which will now carry a G-action.

Recall that if $E \to X$ is a G-equivariant vector bundle then $E^G \to X^G$ is a vector bundle as image of the projection $p_G : E|_{X^G} \to E^G$ given by $p_G(x, v) = (x, \int_G gv\, dg)$, for any $x \in X^G$ and $v \in E_x$.

Proposition 5.5. *Let $\mathcal{I}$ be as in Proposition 4.6, that is let $\mathcal{I}$ be a subalgebra of $\mathcal{C}_0^\infty(X, Y; \mathcal{F})$ containing $\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})$. We endow $\mathcal{I}$ with the C^*-norm of $\mathcal{C}_0(X \smallsetminus Y; \mathcal{F})$ and we complete the cyclic mixed complexes of $\mathcal{I}$, $\mathcal{C}_0^\infty(X, Y; \mathcal{F})$ and $\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})$ with respect to the projective tensor product. Let us assume that $GY = Y$. Assume also that the action of G on X has a single isotropy type H and that $Y \subset \partial X$ is a union of closed faces of Y. Then $\mathcal{I}^G$ is a Connes algebra. In particular, both*

$$\mathcal{C}_0^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_0^\infty(X^H/N(H), Y^H/N(H); \mathcal{F}^H/N(H)) \quad \text{and}$$
$$\mathcal{C}_\infty^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_\infty^\infty(X^H/N(H), Y^H/N(H); \mathcal{F}^H/N(H))$$

are Connes algebras.

Proof. Let $\Gamma := N(H)/H$. The assumptions combined with the diffeomorphism of Equation (13) give $X = X(H) \simeq (G/H) \times_\Gamma X_H = (G/H) \times_\Gamma X^H$, and hence

$$\mathcal{C}_0^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_0^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma \quad \text{and} \quad \mathcal{C}_\infty^\infty(X, Y; \mathcal{F})^G \simeq \mathcal{C}_\infty^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma .$$

Since the action of $\Gamma := N(H)/H$ on X^H is free, the quotient X^H/Γ is also a manifold with corners and $\mathcal{F}^H$ descends to a bundle of algebras $\mathcal{F}^H/\Gamma \to X^H/G$ such that

$$\mathcal{C}_0^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma \simeq \mathcal{C}_0^\infty(X^H/\Gamma, Y^H/\Gamma; \mathcal{F}^H/\Gamma) \quad \text{and}$$
$$\mathcal{C}_\infty^\infty(X^H, Y^H; \mathcal{F}^H)^\Gamma \simeq \mathcal{C}_\infty^\infty(X^H/\Gamma, Y^H/\Gamma; \mathcal{F}^H/\Gamma) .$$

The result then follows from Theorem 4.7 applied to $(X^H/\Gamma, Y^H/\Gamma, \mathcal{F}^H/\Gamma)$ and $\mathcal{I}^G$. □

Note that, even if $\mathcal{F}$ is a *trivial* bundle of semisimple algebras, $\mathcal{F}^H/\Gamma$ need not be trivial, and hence the extra generality afforded by our setting of sections of algebra bundles is necessary.

5.3. **Blow-ups of singular strata.** Recall that Y is a closed submanifold of the compact manifold with corners X and that $[X : Y]$ denotes the blow-up manifold. The definition of the blow-up is recalled in the Appendix in Equation (23). We refer to the Appendix for more on the blow-up. Let $\{H_1, H_2, \ldots, H_N\}$ be a complete set of representatives of conjugacy classes of isotropy groups of G acting on X with admissible ordering. Let

$$V_k := X(H_1) \cup X(H_2) \cup \ldots \cup X(H_k),$$

$k = 1, \ldots, N$, which is a closed subset of X for each k, by Lemma 5.4. Let $\mathcal{F}$ be a G-equivariant bundle of finite dimensional semisimple algebras on X.

Notation 5.6. We let $I_0 := \mathcal{C}_\infty^\infty(X, \partial X; \mathcal{F})$ and $I_k := \mathcal{C}_0^\infty(X, V_k; \mathcal{F}) \cap I_0$, for $k \geq 1$.

In particular,

$$0 = I_N \subset I_{N-1} \subset \ldots \subset I_1 \subset I_0 := \mathcal{C}_\infty^\infty(X, \partial X; \mathcal{F}) .$$

We assume from now on that the action of G on X is boundary intersection free, as in Lemma 5.2. See Equation (23) in the appendix for the definition of the blow-up. We want some sort of "resolution" of the sets V_k. Let H_j be as above (arranged in an admissible order) and let us define by induction X_k and Y_k as follows

$$\begin{cases} X_1 := X , & \\ Y_k := X_k(H_k) , & \text{if } X_k \text{ was defined,} \\ X_{k+1} := [X_k : Y_k] := [X_k : X_k(H_k)] , & \text{if } Y_k \text{ was defined.} \end{cases} \tag{15}$$

Then X_k is a compact manifold with corners with isotropy types $\{H_k, H_{k+1}, \ldots, H_N\}$, $k \leq N$, as in [1]. This procedure works since Y_k is a closed submanifold with corners of X_k by Lemmas 5.1 and 5.2, since H_k is a maximal isotropy group of X_k (see also [1, 25]).

We next want to identify the effect of these blow-ups on the sets $X_{k+1}(H_j)$, $j \geq k + 1$, which we know are manifolds with corners in view of Lemma 5.2 (which the reader should review now since it will be used again below). First of all, the disjoint union decomposition defining X_{k+1} as the blow-up of X_k along its subset

$Y_k := X_k(H_k)$ of points of isotropy of type H_k, namely $X_{k+1} = \big[X_k \smallsetminus Y_k\big] \cup SN^{X_k} Y_k$, gives for $j \geq k+1$

$$X_{k+1}(H_j) = \big[X_k \smallsetminus Y_k\big](H_j) \cup \big[SN^{X_k} Y_k\big](H_j)\,. \tag{16}$$

Since β is a diffeomorphism outside $SN^{X_k} Y_k := SN^{X_k} X_k(H_k)$, we further have

$$\big[X_{k+1} \smallsetminus SN^{X_k} Y_k\big](H_j) \;=\; \big[X_k \smallsetminus Y_k\big](H_j) \;=\; X_k(H_j)\,, \tag{17}$$

since Y_k has isotropy type H_k and $j \geq k+1$. Lemma 5.2 allows us to give an unambigous sense to $\partial X(H) = [\partial X](H) = \partial[X(H)]$ and, together with the fact that $SN^{X_k} Y_k$ is contained in the boundary of X_{k+1} gives that the blow-down map induces for $j \geq k+1$ diffeomorphisms

$$\beta : X_{k+1}(H_j) \smallsetminus \partial X_{k+1}(H_j) \simeq X_k(H_j) \smallsetminus \partial X_k(H_j) \simeq \ldots \simeq X_1(H_j) \smallsetminus \partial X_1(H_j)\,. \tag{18}$$

(Thus, in case $X_1 = X$ does not have a boundary, $Y_k := X_k(H_k)$ will be a compact manifold with corners with interior diffeomorphic to the stratum $X_1(H_k)$, that is, a compactification of $X(H_k)$.)

Recall that $V_k := X(H_1) \cup X(H_2) \cup \ldots \cup X(H_k)$. We obtain then (with $Y_{k+1} := X_{k+1}(H_{k+1})$ and $X_1 = X$, as before) that

$$\beta(Y_{k+1}) \subset V_{k+1} \quad \text{and} \quad \beta(\partial Y_{k+1}) \subset V_k \cup \partial X\,. \tag{19}$$

Consequently, given a smooth section f of $\mathcal{F}$ on V_{k+1} that vanishes on $V_k \cup \partial X$, then $\beta^*(f) := f \circ \beta$ will be a smooth section of (the pull-back of) $\mathcal{F}$ on the compact manifold $Y_{k+1} := X_{k+1}(H_{k+1})$ that vanishes on its boundary. This proves the second inclusion of the following proposition, the proof of the first one (by induction on N) being relegated to the next subsection.

Proposition 5.7. *Assume that the action of G on X is boundary intersection free. Let $\beta : X_{k+1} \to X_1 := X$ be the composition of all the blow-down maps and $J_{k+1} := \beta^*(I_k)/\beta^*(I_{k+1})$. Let $0 \leq k \leq N-1$. Then $I_k/I_{k+1} \to J_{k+1}$ is an isomorphism of topological algebras and*

$$\mathcal{C}^\infty_\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F}) \subset J_{k+1} \subset \mathcal{C}^\infty_0(Y_{k+1}, \partial Y_{k+1}; \mathcal{F})\,.$$

It is also easy to prove that $\mathcal{C}^\infty_c(Y_{k+1}, \partial Y_{k+1}; \mathcal{F}) \subset J_{k+1}$, but we have not been able to use this observation to prove the first inclusion of Proposition 5.7. In any case, this adds credibility to our statement and justifies the postponement of its proof.

Corollary 5.8. *We use the notation and hypotheses of Proposition 5.7 (in particular, the action of G on X is boundary intersection free). Then the algebras J_{k+1} and J^G_{k+1} are Connes algebras.*

Proof. The fact that J_{k+1} is a Connes algebra is a consequence of Theorem 4.7 and Proposition 5.7. Since $Y_{k+1} := X_{k+1}(H_{k+1})$ has a single isotropy type, Proposition 5.5 also yields right away that $\mathcal{C}^\infty_\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F})^G$ is a Connes algebra. Since $J^G_{k+1}/\mathcal{C}^\infty_\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F})^G$ is a topologically nilpotent algebra we obtain that J^G_{k+1} is a Connes algebra as well. ☐

As a corollary, we are now in position to prove the following result.

Theorem 5.9. *We use the notation and hypotheses of Proposition 5.7 (in particular, the action of G on X is boundary intersection free). Then*

(1) The algebras $\mathcal{C}^\infty_\infty(X, \partial X; \mathcal{F})^G$ and $\mathcal{C}^\infty_0(X, \partial X; \mathcal{F})^G$ are Connes algebras.

(2) *Assume furthermore that X has embedded faces and that $Y \subset \partial X$ is a G-invariant union of closed faces of X, then the algebra $\mathcal{C}^\infty_\infty(X, Y; \mathcal{F})^G$ and $\mathcal{C}^\infty_0(X, Y; \mathcal{F})^G$ are Connes algebras.*

Proof. The fact that $\mathcal{C}^\infty_\infty(X, \partial X; \mathcal{F})^G$ is a Connes algebra follows by applying Theorem 3.9 to the stratification (or composition series) I^G_k which has subquotients J^G_{k+1} that are Connes algebras by Corollary 5.8. The fact that $\mathcal{C}^\infty_0(X, \partial X; \mathcal{F})^G$ is a Connes algebra follows from the fact that $\mathcal{C}^\infty_0(X, \partial X; \mathcal{F})^G / \mathcal{C}^\infty_\infty(X, \partial X; \mathcal{F})^G$ is topologically nilpotent. For the second point, notice that $\mathcal{C}^\infty_0(X, Y; \mathcal{F})^G$ has a composition series with subquotients $\mathcal{C}^\infty_0(F, \partial F; \mathcal{F})^G$, where F ranges through a set of closed faces of X. □

5.4. **Proof of Proposition 5.7.** The proof of Proposition 5.7 is an induction on N, the number of isotropy types of X and is a consequence of Lemma 5.10. We shall freely use the notation of the previous subsection and Appendix A (but we also recall some of the most important ones from time to time). Since the result is local, we may assume that $\mathcal{F} = \underline{\mathbb{C}}$, and thus drop it from the notation.

If $N = 1$, $J_1 = I_0 = \mathcal{C}^\infty_\infty(X_1, \partial X_1)$, $Y_1 = X = X_1$ and there is nothing to prove.

Let us now turn to the induction step, by assuming the result to be true if there are $N - 1$ isotropy types and prove it if there are N isotropy types. Let $\{H_1, H_2, \ldots H_N\}$ be the isotropy types of $X =: X_1$. To simplify notation, let $Y := Y_1$, $\tilde{X} := [X_1 : Y_1] = [X : Y_1]$ where, we recall $Y_1 := X(H_1)$. Let $\beta : \tilde{X} \to X = X_1$ be the blow-down map. Similarly, let $\tilde{Y} := \beta^{-1}(Y) := \beta^{-1}(Y_1)$.

For the induction step, we only need to prove that $\mathcal{C}^\infty_\infty(Y_{k+1}, \partial Y_{k+1}) \subset J_{k+1}$ since the inclusion $J_{k+1} \subset \mathcal{C}^\infty_0(Y_{k+1}, \partial Y_{k+1})$ is obvious (and was already discussed). Let us consider the composition series of 5.6 for $\tilde{X}$, but we shift the indices to account for the fact that $\tilde{X}$ has only $N - 1$ isotropy types:

$$\tilde{I}_k := \mathcal{C}^\infty_0(\tilde{X}, \tilde{Y}_k) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \partial\tilde{X}), \quad k \geq 2, \tag{20}$$

and $\tilde{I}_1 := \mathcal{C}^\infty_\infty(\tilde{X}, \partial\tilde{X})$. (Thus the definition of I_k is obtained from the definition of $\tilde{I}_k$ by removing all the symbols $\tilde{\ }$.)

Recall from Lemma A.1 that the pull back $\beta^*(f) := f \circ \beta$ defines an isomorphism $\beta^* : \mathcal{C}_0(X \smallsetminus Y) \to \mathcal{C}_0(\tilde{X} \smallsetminus \tilde{Y})$. Moreover, Lemma A.3 shows that it also defines an isomorphism

$$\beta^* : \mathcal{C}^\infty_\infty(X, Y) \longrightarrow \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}).$$

To complete the proof, we shall need the following result, which is best formulated as a lemma.

Lemma 5.10. *Let $\beta^{*-1} : \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}) \to \mathcal{C}^\infty_\infty(X, Y)$ be the map of Lemma A.3. Then $\beta^{*-1}(\tilde{I}_k) \subset I_k$, $k = 1, \ldots, N$.*

Proof. We have that $\partial\tilde{X} = \tilde{Y} \cup \beta^{-1}(\partial X)$. Hence $\tilde{I}_1 := \mathcal{C}^\infty_\infty(\tilde{X}, \partial\tilde{X}) = \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \beta^{-1}(\partial X))$. Equation (18) gives

$$\tilde{I}_k := \mathcal{C}^\infty_0(\tilde{X}, \tilde{Y}_k) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \tilde{Y}) \cap \mathcal{C}^\infty_\infty(\tilde{X}, \beta^{-1}(\partial X)), \tag{21}$$

for all k (that is, $k = 1, \ldots, N$). Let then $f \in \tilde{I}_k$. By Lemma A.3, $\beta^{*-1}(f) \in \mathcal{C}^\infty_\infty(X, Y)$. It is obvious that $\beta^{*-1}(f)$ also vanishes to infinite order on ∂X since $Y = Y_1$ is an (interior) submanifold with corners (so $\partial X \smallsetminus Y$ is dense in ∂X). It follows that $\beta^{*-1}(f)$ also vanishes on Y_k since f vanishes (even of infinite order) on $\tilde{Y}_1$ and it vanishes on $\tilde{Y}_k \smallsetminus \tilde{Y}_1$, which is mapped bijectively onto its image by β. Hence $\beta^{*-1}(f) \in \mathcal{C}^\infty_\infty(X,Y) \cap \mathcal{C}^\infty_0(X,Y_k) =: I_k$. □

We can now complete the proof. By the induction hypothesis (which gives the first inclusion in the next displayed equation) and since the space Y_{k+1} is the same for both X and $\tilde{X}$ (by construction), we have

$$\mathcal{C}^\infty_\infty(Y_{k+1}, \partial Y_{k+1}; \mathcal{F}) \subset \tilde{I}_k/\tilde{I}_{k+1} \xrightarrow{\beta^{*-1}} I_k/I_{k+1}\,. \tag{22}$$

6. Further applications

We now include a few direct applications of our previous results.

6.1. **Crossed product with finite groups.** We keep the notations of the previous subsections. Then we obtain the following result (see also [8, 36]):

Theorem 6.1. *Assume that G is a finite group. Let X be a compact, boundary intersection free G-manifold with corners and let $\mathcal{E} \to X$ be a G-algebra bundle over X with simple algebra fibers and let $Y \subset \partial X$ be a G-invariant union of closed faces of X. Then the crossed product algebra $\mathcal{C}^\infty(X,Y;\mathcal{E}) \rtimes G$ is a Connes Fréchet m-algebra that is spectrally invariant in its C^*-completion $\mathcal{C}_0(X \smallsetminus Y;\mathcal{E}) \rtimes G$. In particular, the Chern-Connes-Karoubi character induces an isomorphism*

$$\widetilde{\mathrm{Ch}} : K_n^{\mathrm{top}}\left(\mathcal{C}_0(X \smallsetminus Y, \mathcal{E}) \rtimes G\right) \xrightarrow{\simeq} \mathrm{HP}_n^{\mathrm{top}}\left(\mathcal{C}^\infty(X,Y;\mathcal{E}) \rtimes G\right)\,, \quad n = 0, 1\,.$$

Proof. We denote $\mathcal{A} := \mathcal{C}^\infty(X,Y,\mathcal{E})$, $A = \mathcal{C}_0(X \smallsetminus Y;\mathcal{E})$, $\mathcal{B} := \mathcal{A} \rtimes G$ and $B := A \rtimes G$. The algebra $\mathcal{B}$ can be identified with the algebra $(\mathcal{A} \otimes \mathrm{End}(V))^G$ where V is the finite dimensional G-representation on $V = \ell^2 G$ (the regular representation). In the same way, the algebra B can be identified with the C^*-algebra $(A \otimes \mathrm{End}(V))^G$. Indeed, an algebra isomorphism is obtained by identifying any kernel $k : G \times G \to \mathcal{A}$, which is G-equivariant, with a function of a single variable in G, and it is easy to check that this is a topological identification for the smooth functions and a C^*-algebra isomorphism for the completions. More precisely, the isomorphism is given by the map $\Phi : (\mathcal{A} \otimes \mathbb{C}[G \times G])^G \to \mathcal{A} \rtimes G$ defined for $k \in (\mathcal{A} \otimes \mathbb{C}[G \times G])^G$ by $\Phi(k)(g) = k(e,g)$, where $e \in G$ denotes the identity element. Since $k(h,g) = h(k(e,h^{-1}g))$ by G-invariance, we clearly get

$$\begin{aligned}\Phi(k{\cdot}k')(g) = \sum_{h\in G} k(e,h)k'(h,g) = \sum_{h\in G} k(e,h)h(k'(e,h^{-1}g)) &= \sum_{h\in G} \Phi(k)(h)h(\Phi(k')(h^{-1}g)) \\ &= \Phi(k) * \Phi(k')(g).\end{aligned}$$

The inverse map is given by $\Phi^{-1}(f)(g,h) = g(f(g^{-1}h))$ which is clearly G-invariant because

$$u(\Phi^{-1}(f)(u^{-1}g, u^{-1}h)) = g(f(g^{-1}h)) = \Phi^{-1}(f)(g,h).$$

Similar computations give $\Phi^{-1}(f_1 * f_2) = \Phi^{-1}(f_1) \cdot \Phi^{-1}(f_2)$. Therefore, it remains to show that the Fréchet m-algebra $(\mathcal{A} \otimes \mathrm{End}(V))^G$ is a Connes algebra which is spectrally invariant in its C^*-completion $(A \otimes \mathrm{End}(V))^G$. Now, this is a consequence of Theorem 5.9. □

6.2. **Pseudodifferential operators.** Assume now that M is a compact, *smooth* manifold (so without corners). Let $B^*M \subset T^*M$ be the set of vector of length ≤ 1. Then there is an identification $S^0_{cl}(T^*M) \simeq \mathcal{C}^\infty(B^*M)$ using a suitable compactification of T^*M. There exists also a "quantization map" $q : S^0_{cl}(T^*M) \to \Psi^0(M)$ and an isomorphism $\mathcal{R} \simeq \Psi^{-\infty}(M)$ such that the induced map $\mathcal{C}^\infty(B^*M) \oplus \mathcal{R} \to \Psi^0(M)$ is onto. We endow $\Psi^0(M)$ with the induced Fréchet topology that makes it an m-algebra. We define similarly the topology on algebras of families of order zero pseudodifferential operators.

6.2.1. *Isotypical components.* We shall denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of bounded operators over a Hilbert space $\mathcal{H}$. Let as before $E \to M$ be a G-equivariant hermitian vector bundle. Let α be an irreducible unitary representation of G and denote by $L^2(M,E)_\alpha \cong \alpha \otimes L^2(M, E \otimes \alpha^*)^G$ the isotypical component associated with α. Let us consider the restriction map $\pi_\alpha : \mathcal{B}(L^2(M,E))^G \to \mathcal{B}(L^2(M,E)_\alpha)^G$, see [5, 6, 7]. Recall that we have $\mathcal{K}(L^2(M,E))^G = \bigoplus \mathcal{K}(L^2(M,E)_\alpha)^G$ and $\mathcal{K}(L^2(M,E)_\alpha)^G \cong \mathcal{K}(L^2(M, E \otimes \alpha^*)^G)$, see [5]. We get then the following result.

Proposition 6.2. *The algebra $\pi_\alpha(\Psi^{-\infty}(M,E)^G)$ is a Connes algebra. If G is finite, then $\Psi^{-\infty}(M,E)^G$ is also a Connes algebra.*

Proof. If α does not appear in $L^2(M,E)$ then $\pi_\alpha(\Psi^{-\infty}(M,E)^G) = 0 = \mathcal{K}(L^2(M,E)_\alpha)^G$ and this is clearly a Connes algebra. Assume now that α appears in $L^2(M,E)$. We have $\pi_\alpha(\Psi^{-\infty}(M,E)^G) = \Psi^{-\infty}(M,E)^G \cap \mathcal{K}(L^2(M,E)_\alpha)^G \cong \mathcal{R}$. Thus the result follows. □

A similar proof to Theorem 5.9 (using also the argument in the proof of Proposition 6.3) gives also that $\pi_\alpha(\Psi^{-\infty}(M,E)^G)$ is a Connes algebra as well (this result is one of the main motivation for this paper). The proof will be included in a forthcoming publication, in order not increase the lenght of this paper too much.

6.2.2. *Families of operators.* Let $p : M \to B$ be a locally trivial fibration of riemannian compact manifolds without boundary. Let $\pi : E \to M$ be a vector bundle. Denote by $\Psi^m(M|B,E)$ the set of smooth families $P = (P_b)_{b \in B}$ of pseudodifferential operators. Denote by $S^*(M|B) \to M$ the cosphere bundle of the vertical tangent bundle $T(M|B) = \ker dp$.

Proposition 6.3. *The algebras $\mathcal{C}^\infty(S^*(M|B), \mathrm{End}(E))$, $\Psi^{-\infty}(M|B,E)$, $\Psi^{-1}(M|B,E)$ and $\Psi^0(M|B,E)$ are Connes algebras.*

Proof. This is completely similar to the case of one operator, see Proposition 6.2. The result for the first algebra $\mathcal{C}^\infty(S^*(M|B), \mathrm{End}(E))$ is clear. The result for the third algebra $\Psi^{-1}(M|B,E)$ follows from the fact that $\Psi^{-1}(M|B,E)/\Psi^{-\infty}(M|B,E)$ is topologically nilpotent, using Proposition 4.2, and the proof that the second algebra is a Connes algebra given below. The result for the last algebra is then a consequence of Theorem 3.9 using the exact sequence

$$0 \to \Psi^{-1}(M|B,E) \to \Psi^0(M|B,E) \to \mathcal{C}^\infty(S^*(M|B), \mathrm{End}(E)) \to 0.$$

To finish the proof notice that $\Psi^{-\infty}(M|B,E) \cong \mathcal{R}\mathcal{C}^\infty(B)$. Indeed, for each b, the operator P_b is isomorphic to an element $(m_{ij}(b)) \in \mathcal{R}$ and this depends smoothly on b as can be checked on any trivialisation U of $p : M \to B$ because $\Psi^{-\infty}(M|B,E)|_U \cong \mathcal{C}^\infty(U, \Psi^{-\infty}(F,E'))$, where F is the typical fiber of $p : M \to B$ and E' the typical fiber of $p \circ \pi : E \to B$. □

In particular, let M be a compact, *smooth* manifold (so without corners) and $E \to M$ be a vector bundle and Γ be a finite group acting on M and E. Then the algebra $\Psi^0(M;E)^\Gamma$ is a Connes algebra. Indeed, Proposition 6.2 shows that $\Psi^{-\infty}(M,E)^\Gamma$ is a Connes algebra therefore $\Psi^{-1}(M,E)^\Gamma$ is a Connes algebra because the quotient is topologically nilpotent and $\mathcal{C}^\infty(S^*M, \mathrm{End}(E))^\Gamma$ is a Connes algebra by Theorem 5.9 applied with $X = S^*M$, $\partial X = \emptyset$ and $G = \Gamma$. Thus Theorem 1.2 applies to the exact sequence

$$0 \to \Psi^{-1}(M,E)^\Gamma \to \Psi^0(M,E)^\Gamma \to \mathcal{C}^\infty(S^*M, \mathrm{End}(E))^\Gamma \to 0.$$

and gives that $\Psi^0(M,E)^\Gamma$ is a Connes algebra. However, if Γ is infinite, this result is not true anymore, one needs to consider a weaker property, thus leading to "weak Connes algebras," for which we only require the Chern-Connes-Karoubi character to be injective with dense image. Indeed, take $M = G$ a non discrete compact Lie group, (for instance $S^1 = \{z \in \mathbb{C},\ |z| = 1\}$). Then $\Psi^{-\infty}(G)^G \cong \mathcal{C}^\infty(G)$ and therefore $RK_j(\Psi^{-\infty}(G)^G) \cong RK_j(\Psi^{-1}(G)^G) \cong K_j(C^*G)$, where $C^*G \cong \mathcal{K}(L^2(G))^G$ is the C^*-algebra of the group G. But $K_0(C^*G) = R(G)$ is the representation ring of G and $K_1(C^*G) = 0$. It is well known that $R(G) = \bigoplus_{\hat{G}} \mathbb{Z}$, where $\hat{G}$ is the set of isomorphism classes of irreducible unitary representations of G. Moreover, $HP_*(\mathcal{C}^\infty(G)) = \mathcal{C}^\infty(G)^G$, see [32, 29] for instance and thus the Connes' character has only dense image. Now since $\mathcal{C}^\infty(S^*G)^G$ is a Connes algebra, we get that $\Psi^0(G)^G$ can not be a Connes algebra. See also the Appendix of the second (enhanced) version of [36], available from the author's home page.

Appendix A. Smooth functions and blow-ups

Recall that if X is a manifold with corners and $Y \subset X$ is a submanifold with corners, then the normal bundle N^XY of Y in X is diffeomorphic to an open neighborhood U of Y in X by a diffeomorphism ϕ that maps the zero section of the normal bundle N^XY to Y. Let SN^XY be the unit sphere bundle of N^XY. (So its fibers are spheres, as the name indicates it.) Then [27, 3] the *blow-up of X along Y* (or with respect to Y) is the *disjoint union*

$$[X : Y] := (X \smallsetminus Y) \cup SN^XY\,. \tag{23}$$

It comes equipped with the structure of a smooth manifold with corners and a *blow-down* map $\beta : [X : Y] \to X$ that is the identity on $X \smallsetminus Y$ and is the bundle projection $SN^XY \to Y$ on SN^XY. We shall identify without further comment $X \smallsetminus Y$ with $[X : Y] \smallsetminus \beta^{-1}(Y)$ in what follows. Assume that a Lie group Γ acts smoothly on X such that $\Gamma Y = Y$. Then the action of Γ on X lifts to a smooth action of Γ on $[X : Y]$ (this was proved in the case G compact in [1] and in general in [3]).

Let us denote $[X : Y]$ by $\tilde{X}$ and set $\tilde{Y} := \beta^{-1}(Y)$.

Lemma A.1. *The pull back $\beta^*(f) := f \circ \beta$ defines an isomorphism $\beta^* : \mathcal{C}_0(X \smallsetminus Y) \to \mathcal{C}_0(\tilde{X} \smallsetminus \tilde{Y})$.*

Proof. This is because the blow-down map $\beta : \tilde{X} \to X$ is such that X has the quotient topology and $\beta(\tilde{Y}) = Y$. □

Let us denote by $\mathrm{Diff}(X)$, respectively, $\mathrm{Diff}(\tilde{X})$ the *algebra of differential operators* with smooth coefficients on X, respectively, $\tilde{X}$. Let r_Y be a smooth function on X such that, close to Y it is the distance to Y and otherwise it is > 0 outside

Y. Then it is known that r_Y lifts to a smooth function $r_Y \circ \beta$ on $\tilde{X}$ whose zero set is exactly $\tilde{Y}$. Notice that

$$\begin{aligned} \mathcal{C}^\infty_\infty(X,Y) &:= \{u \in \mathcal{C}_0(X \smallsetminus Y) \mid Pu \in \mathcal{C}_0(X \smallsetminus Y) \ \text{ for all } P \in \mathrm{Diff}(X)\}\,, \\ \mathcal{C}^\infty_\infty(\tilde{X},\tilde{Y}) &:= \{u \in \mathcal{C}_0(\tilde{X},\tilde{Y}) \mid Pu \in \mathcal{C}_0(\tilde{X} \smallsetminus \tilde{Y}) \ \text{ for all } \tilde{P} \in \mathrm{Diff}(\tilde{X})\}\,, \\ & r_Y^{-1}\mathcal{C}^\infty_\infty(X,Y) \subset \mathcal{C}^\infty_\infty(X,Y)\,, \quad \text{and} \\ & r_Y^{-1}\mathcal{C}^\infty_\infty(\tilde{X},\tilde{Y}) \subset \mathcal{C}^\infty_\infty(\tilde{X},\tilde{Y})\,. \end{aligned} \tag{24}$$

Lemma A.2. *We have the following equalities as operators on* $\mathcal{C}^\infty_c(X \smallsetminus Y) = \mathcal{C}^\infty_c(\tilde{X} \smallsetminus \tilde{Y})$*:*

(1) $\mathrm{Diff}(X) \subset \cup_{k=1}^\infty r_Y^{-k}\,\mathrm{Diff}(\tilde{X})$ *and, similarly,*
(2) $\mathrm{Diff}(\tilde{X}) \subset \cup_{k=1}^\infty (r_Y \circ \beta)^{-k}\mathcal{C}^\infty(\tilde{X})\,\mathrm{Diff}(X)$.

Proof. This is a standard fact about blow-ups. For the first relation, since r_Y is smooth on $\tilde{X}$, $\cup_{k\in\mathbb{N}} r_Y^{-k}\,\mathrm{Diff}(\tilde{X})$ is an algebra. It is enough then to prove our statement for a system of generators of $\mathrm{Diff}(X)$. This is clear if P is a multiplication operator, since a smooth function on X lifts to a smooth function on $\tilde{X}$. Let v be a vector field on X. Then it is known (see, for instance [3]) that there exists a vector field $\tilde{v}$ on $\tilde{X}$ that restricts to v in the interior. Hence $\tilde{v} \in \mathrm{Diff}(\tilde{X})$ and $v = r_Y \circ \beta^{-1}\tilde{v} \in \cup_{k\in\mathbb{N}} r_Y^{-k}\,\mathrm{Diff}(\tilde{X})$, as desired.

It is easy to see in local coordinates that the algebra $\mathrm{Diff}(\tilde{X})$ is generated by $\mathcal{C}^\infty(\tilde{X})$, $(r_Y \circ \beta)^{-1}$, and the lifts of vector fields $r_Y v$, with v a vector field on X. □

We now come to the following lemma which is used in the proof of Proposition 5.7, but which is obviously of independent interest.

Lemma A.3. *The pull back* $\beta^*(f) := f \circ \beta$ *defines an isomorphism*

$$\beta^* : \mathcal{C}^\infty_\infty(X,Y) \to \mathcal{C}^\infty_\infty(\tilde{X},\tilde{Y})\,.$$

Proof. This follows from the previous two lemmas and Equation (24). Indeed, let $f \in \mathcal{C}^\infty_\infty(X,Y)$ and $\tilde{P} \in \mathrm{Diff}(\tilde{X})$. We want to prove that $\tilde{P}(f \circ \beta) \in \mathcal{C}_0(\tilde{X} \smallsetminus \tilde{Y})$. Then, by the second part of Lemma A.2 we may assume that $\tilde{P} = a r_Y^{-k} P$, with $P \in \mathrm{Diff}(X)$ and $a \in \mathcal{C}^\infty(\tilde{X})$. Then $r_Y^{-k} Pf \in \mathcal{C}^\infty_\infty(X,Y)$ and hence $\tilde{P}(f \circ \beta) = a(r_Y^{-k} Pf) \circ \beta \in \mathcal{C}_0(\tilde{X} \smallsetminus \tilde{Y})$, by Lemma A.1. This shows that the map $\beta^* : \mathcal{C}^\infty_\infty(X,Y) \to \mathcal{C}^\infty_\infty(\tilde{X},\tilde{Y})$ is well defined. It is obviusly injective since $X \smallsetminus Y$ is dense in both X and $\tilde{X}$. Let us prove that it is onto. Let $g \in \mathcal{C}^\infty_\infty(\tilde{X},\tilde{Y})$. Then $g = f \circ \beta$ with some $f \in \mathcal{C}_0(X \smallsetminus Y)$, again by lemma A.1. Let $P \in \mathrm{Diff}(X)$. We similarly want to prove that $Pf \in \mathcal{C}_0(X \smallsetminus Y)$. We have that $P = r_Y^{-k} Q$ for some $Q \in \mathrm{Diff}(\tilde{X})$. Then $Pf \circ \beta = r_Y^{-k} Qg$. Since $r_Y^{-k} Qg \in \mathcal{C}_0(\tilde{X} \smallsetminus \tilde{Y})$ by the assumption on g, we have that $Pf \in \mathcal{C}_0(X \smallsetminus Y)$, again by Lemma A.1. □

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